Algebraic invariants for filtered spaces and their computation

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Abstract

This thesis explores the construction and computation of algebraic invariants for filtered topological spaces. These invariants are derived from the classical algebraic object of homology, which is invariant under homeomorphisms, i.e., continuous transformations of topological spaces with continuous inverses. Filtered spaces, however, are richer structures that often encode geometric information, and thus our algebraic constructions are invariant with respect to isometries and scaling operations. We are also interested in the stability of invariants with respect to small perturbations of the input data, including the addition of noise.

Given a space with a poset-valued filtration function, the homology (with field coefficients) of sublevel sets forms a functor from the poset to vector spaces: this functor is called a *persistence module*, and is a central object in the field of topological data analysis (TDA). In Paper A, we develop the theory of relative homological algebra for persistence modules, including the use of *local Koszul complexes* as a tool for computing *relative Betti diagrams*, which are a numerical invariant of minimal relative projective resolutions. In Papers B and C, we study the case where the indexing poset is a total order. In this case, persistence modules decompose as sums of simple *bar modules*, and we introduce the notion of *bar-to-bar morphisms* between persistence modules as an algebraic version of bar matchings. Moreover, we develop algorithms for computing matchings induced by morphisms, passing through bar-to-bar

morphisms.

Papers D and E generalize a more classical invariant, the critical points of a Morse function, to the setting of *metric algebraic geometry*, an emerging field that combines (or reunites) algebraic and differential geometry. Specifically, we develop a Morse theory for distance functions from an algebraic variety, restricted to an algebraic variety, and show that, generically, such a distance function is Morse. We define both geometric and algebraic notions of critical points and provide upper bounds on their number. In this sense, we construct and compute invariants for filtered algebraic spaces.

Sammanfattning

Denna avhandling utforskar konstruktionen och beräkningen av algebraiska invarianter för filtrerade topologiska rum. Dessa invarianter härstammar från det klassiska algebraiska objektet, homologi, som är invariant under homeomorfismer, d.v.s. kontinuerliga transformationer av topologiska rum med kontinuerliga inverser. Filtrerade rum är emellertid rikare strukturer som ofta kodar geometrisk information, och därför är våra algebraiska konstruktioner invarianta med avseende på isometrier och skalningsoperationer. Vi är också intresserade av stabiliteten hos invarianter med avseende på små störningar i indatan, inklusive tillägg av brus.

Givet ett rum med en pomängdvärderad filtreringsfunktion, bildar homologin (med fältkoefficienter) hos delnivåmängder en funktor från pomängden till vektorrummen: denna funktor kallas en *persistensmodul* och är ett centralt objekt inom området topologisk dataanalys (TDA). I Artikel A utvecklar vi teorin om relativ homologisk algebra för persistensmoduler, inklusive användningen av *lokala Koszulkomplex* som ett verktyg för att beräkna *relativa Bettidiagram*, vilka är en numerisk invariant av minimala relativa projektiva upplösningar. I Artiklarna B och C studerar vi fallet där indexeringspomängden är en total ordning. I detta fall sönderfaller persistensmoduler som summor av enkla *streckmoduler*, och vi introducerar begreppet *streck-till-streck-morfismer* mellan persistensmoduler som en algebraisk version av streckmatchningar. Dessutom utvecklar vi algoritmer för att beräkna matchningar inducerade av

morfismer, som passerar genom streck-till-streck-morfismer.

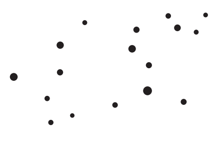
Artiklar D och E generaliserar en mer klassisk invariant, de kritiska punkterna för en Morsefunktion, till *metrisk algebraisk geometri*, ett framväxande område som kombinerar (eller återförenar) algebraisk och differentialgeometri. Mer specifikt utvecklar vi en Morseteori för avståndsfunktioner från en algebraisk varietet, begränsad till en algebraisk varietet, och visar att en sådan avståndsfunktion generellt är Morse. Vi definierar både geometriska och algebraiska begrepp för kritiska punkter och anger övre gränser för deras antal. I denna mening konstruerar och beräknar vi invarianter för filtrerade algebraiska rum.

Popular science summary

Consider the set of reference points in the bottom right corner of this page and define the function that to every other point on the page associates the distance from it to its closest reference point[1]. This defines a *filtered space*, where the space is the page and the filtration is the distance function. The set of points with distance at most some threshold value is called a *sublevel set*. Starting from 0, we can increase the threshold value progressively, thus obtaining a sequence of sublevel sets of the plane. Flip through the pages of this thesis to see successive sublevel sets outlined in black (the additional shapes represent a *simplicial complex*, see Definition 2.2).

We are interested in studying how the *topology*, or shape, of these sublevel sets evolves over time: clusters can merge, loops can appear and disappear, and so on. This study is central to the two fields covered by this thesis, *Morse theory* and *topological data analysis* (TDA). Both fields use tools called *algebraic invariants* to summarize the changes in topology. While there are an infinite number of sublevel sets — one for each threshold value —, there are typically only finitely many changes in topology. This finiteness of topological changes translates to finite-sized invariants: for example, a finite number of *critical points* (for Morse theory) or a finite *barcode* (for TDA). This leads to the question of computing algebraic invariants, which this thesis also addresses. While

[1] For example, consider the dot on the *i* in *point*. Its closest reference point is the leftmost one, and the distance to it is around 49 mm.



Morse theory focuses more on formal computations, such as bounding the number of topological features by the number of critical points, TDA's success is due to the computability of its invariants.

The results of this thesis can be sorted into three topics, with the first two pertaining to TDA and the last to Morse theory. In Paper A, we define and compute an algebraic invariant called a *relative Betti diagram* for a generalized form of filtered space, where the filtration's values belong to an arbitrary partially ordered set instead of real numbers. In Papers B and C, we stick to standard filtrations and study how functions between filtered spaces induce matchings between barcodes. In Papers D and E, we define Morse theory for spaces filtered by distance functions (the original theory is for smoother filtration functions). In particular, we study distance functions between spaces called *algebraic varieties*, where we show that, in general, the Morse theory behaves well: critical points capture changes in topology and there are finitely many of them.

The study of algebraic invariants for filtered spaces is a rich topic in mathematics at the intersection of theory and computation. This thesis provides a glimpse into the various questions that interest researchers, and opens some new directions to explore.

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Jens Agerberg, Andrea Guidolin, Isaac Ren, and Martina Scolamiero. In *Journal of Applied and Computational Topology*, 2025. DOI: `10.1007/s41468-024-00200-w`

C Barcode matchings and p-norms of persistence modules 181
Andrea Guidolin, Arthur Reidenbach, Isaac Ren, and Martina Scolamiero. To be submitted, 2026.

D Morse theory of Euclidean distance functions from algebraic hypersurfaces 225
Andrea Guidolin, Antonio Lerario, Isaac Ren, and Martina Scolamiero. Submitted, 2025. arXiv: `2402.08639 [math.AG]`

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Abbreviations and notations

Abbreviations

BND	bottleneck degree
CI	complete intersection
EDD	Euclidean distance degree
p.f.d.	pointwise finite-dimensional
TDA	Topological data analysis

Notations

Symbol	Description
a, b	element of poset
$B(x, r)$	closed ball of center x and radius r
Cat	category
d	simplex dimension, homological degree, (pseudo)metric, polynomial degree
∂	boundary map of chain complex
f, g, h	real-valued functions
F, G	functor
i, j, k	index
I, J	poset

Symbol	Description
K	simplicial complex
$\mathbf{k}$	underlying field for vector spaces
M	matching of sets or multisets
$\mathbf{N}$	set or poset of natural numbers
$\mathbf{R}$	set or poset of real numbers
s, t	element of poset; start- and end-point of an interval
S, T	subset of poset
v	vertex
U, V, W	persistence module
x, y	element of space
X, Y	topological space
α	index
λ	element of $\mathbf{k}$
σ, τ	simplices

A note on reading this thesis

Defined terms are printed in **boldface** and are searchable in the index at the back of this thesis. Relevant notations are also listed in the index. The ends of numbered passages, such as definitions, examples, and remarks, are marked by a black square (■), as opposed to the ends of proofs, which are marked by a white square (□).

The notations used in the *kappa* sometimes differ from those of the included papers, as it was impossible to harmonize all notations while avoiding ambiguities. Notably, persistence modules here are denoted by V, W, and so on, while they are denoted by F, M, etc. in Paper A and by X, Y, etc. in Papers B and C.

Background and results

1 Introduction

I was of three minds,
Like a tree
in which there are three blackbirds.[1]

This thesis consists of three parts: the front matter, which you have just passed, the *kappa*, which you are reading, and the included papers, which you will get to shortly. This thesis is also the outcome of approximately three research questions: one about computing relative homological algebra for poset-indexed functors, another about inducing barcode matchings from morphisms of persistence modules, and a third about generalizing Morse theory to distance functions between algebraic varieties. All of these questions, their answers, and the resulting further questions will be presented in this *kappa*. Most importantly, this thesis revolves around three mathematical trios: filtered topological, algebraic, and combinatorial spaces; algebraic-topological, -geometric, and -persistent invariants; and generic, algorithmic, and numeric computations.

There is no one-to-one correspondence between the research questions and the mathematical concepts; each project involves each concept to some extent. However, the results of this thesis can be split into those studying persistence modules and those studying Morse theory.

---

[1]The chapter epigraphs are stanzas II, XII, IX, III, and XIII respectively, from the poem "Thirteen Ways of Looking at a Blackbird" by Wallace Stevens (1879-1955) [47].

In the words of Gunnar Carlsson, a foundational researcher in **topological data analysis** (**TDA**), “data has shape and the shape matters” [14]. TDA is a relatively new field, established around the turn of the 21st century, that applies invariants from topology and algebraic topology to understand the topological and geometric structure of data. One of the central objects in TDA, which quantifies the shape of, and encodes information about, data is the **persistence module**, which is the image of a homology functor applied to a filtered topological space. In Chapter 2, we present **persistent algebra**, the study of persistence modules, which will be relevant for Papers A, B, and C.

While the first two-thirds of this thesis revolve around persistence modules, the last third studies filtered spaces form a different angle, namely that of **Morse theory**. Morse theory was first developed by the eponymous mathematician under the name *calculus of variations in the large* [40]. In Chapter 3 we give a modern presentation of the original theory for smooth manifolds, as well as its extensions to continuous selections, which are the basis for Papers D and E.

In Chapter 4 we summarize the results of each included paper, and in Chapter 5 we present some ongoing work and future directions of research.

2 Persistent algebra

The river is moving.
The blackbird must be flying.

Persistent algebra is my name for the study of persistence modules using linear and homological algebra: see [12, 13, 33, 37, 50] for examples of this study in the literature. This is a setting where the theory is at times familiar: persistence modules often behave like vector spaces (Section 2.2), and their homological theory is also quite simple (Section 2.3). Nonetheless, there are subtleties that require attention and care, including the proper definition of persistence modules (Section 2.1). The aim of this chapter is to showcase the theory of persistent algebra as it will be used in the included papers.

2.1 Ten Ways of Looking at a Persistence Module

If you are familiar with TDA, then you may be aware of the perhaps excessive number of resources introducing the field and its central tool, persistent homology [17, 20, 25, 41, 42, and the "Preliminaries" section of many papers]. This section is yet another presentation of TDA, through ten perspectives. The aim here is not to be exhaustive, but instead to showcase the multifaceted nature of persistence modules. You may skip to Section 2.2 and return as needed for exact definitions and notational conventions.

If you are not familiar with TDA, then read on. In this section, we will explore some of the field's definitions, characterizations, generalizations, and invariants. Along the way, we will cover key concepts in the so-called persistent homology pipeline, including the barcode decomposition (Theorem 2.23), bottleneck stability (Theorem 2.37), and the rank invariant (Definition 2.38).

2.1.1 As persistent homology

In TDA, persistence modules first appeared in the form of **persistent homology**, an algebraic invariant for filtered topological spaces:

2.1 **Definition.** A **filtered space** is a topological space X equipped with a continuous real-valued function $f\colon X \to \mathbf{R}$, called a **filtration**. Given a filtered space (X, f) and a real number $t \in \mathbf{R}$, the **sublevel set of** X **at level** t is the set

$$X_{\leq t} := f^{-1}((-\infty, t]).$$

Let (X, f) be a filtered space. Fix $\mathbf{k}$ a field and $d \geq 0$ an integer. The $d^{\mathbf{th}}$ **persistent homology of** (X, f) **with coefficients in** $\mathbf{k}$ is the collection of

1. homology (vector) spaces $H_d(X_{\leq t}; \mathbf{k})$ for all t in $\mathbf{R}$, and
2. linear maps $H_d(X_{s \leq t}; \mathbf{k})\colon H_d(X_{\leq s}; \mathbf{k}) \to H_d(X_{\leq t}; \mathbf{k})$ induced by the inclusion maps $X_{\leq s} \hookrightarrow X_{\leq t}$ for all $s \leq t$ in $\mathbf{R}$.

We often omit the coefficient field, writing $H_d(X_{\leq t})$ and the like. ■

In this section, we will detail what exactly this definition means in the case of simplicial complexes, computing simplicial homology. First, some motivation for studying simplicial complexes.

A typical persistent homology pipeline begins by viewing a point cloud as a topological space X. This space X is then equipped with one or several filtrations. For example, a subset X of a Euclidean space $\mathbf{R}^n$ can be equipped with the filtrations that project onto each coordinate, as a kinds of height maps: see Figure 3.1 for such an example. Sublevel sets are of interest for tracking topological changes in the data (see Morse theory in Section 3.1), but they are somewhat trivial when the underlying space is discrete, as is the case for finite point clouds. Instead, the main example of filtered spaces in TDA is filtered simplicial complexes.

2.2 **Definition.** An (**abstract**) **simplicial complex** is a set of sets K such that, for every element $\sigma \in K$ and subset $\tau \subseteq \sigma$, τ is also an element of K. An element σ of K with size $|\sigma| = d+1$ is called a d**-simplex**, and we say that σ is of **dimension** d. A **face** of a simplex σ is a subsimplex $\tau \subseteq \sigma$. Conversely, a **coface** of a simplex σ is a supersimplex $\tau \supseteq \sigma$. For all $d \geq 0$, we denote by K_d the set of d-simplices of K. Similarly to graphs, the 0-simplices are called **vertices**, and the 1-simplices are called **edges**.

Let K and L be simplicial complexes. A **morphism of simplicial complexes** $f\colon K \to L$ is a set-theoretic function that is monotone with respect to the face relation: for $\tau \subseteq \sigma$ in K, we requires that $f(\tau) \subseteq f(\sigma)$.

A **filtration** on a simplicial complex K is a real-valued function $f\colon K \to \mathbf{R}$ such that, for all $\tau \subseteq \sigma$ in K, $f(\tau) \leq f(\sigma)$. The **sublevel sets** of K are then $K_{\leq t} := \{\sigma \in K \mid f(\sigma) \leq t\}$ for all $t \in \mathbf{R}$. ■

2.3 **Example.** Given a set X, the **full simplicial complex on** X is $\mathscr{P}(X)$, the set of all subsets of X. This is indeed a simplicial complex, whose vertices are exactly X (up to identification of singletons with their only element). ■

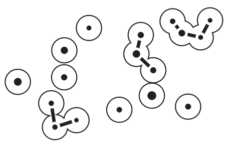
2.4 **Remark.** It is not immediate from the above definition that simplicial complexes are topological spaces. To make this clearer, we view K as a poset for the inclusion relation and equip it with the Alexandrov topology (Definition 2.18). We can show that a continuous map for this topology is exactly a monotone map with respect to the face relation. We will not be using this topology. ■

We now present some typical constructions for filtered simplicial complexes in TDA.

2.5 **Example.** Let (M,d) be a metric space, X a subset of M, and $t \geq 0$ a positive number. The **Čech complex at radius** t is the simplicial complex consisting of all subsets σ of X such that there exists a closed ball $B(x,t)$ of radius t in M containing σ. We denote this complex by $\check{\mathrm{C}}_t(X)$:

$$\check{\mathrm{C}}_t(X) := \left\{ \sigma \subseteq X \;\middle|\; \bigcap_{x\in\sigma} B(x,t) \neq \varnothing \right\}.$$

The **Vietoris-Rips complex at radius** t is the simplicial complex consisting of all subsets σ of X that have pairwise distance at most t. We denote this



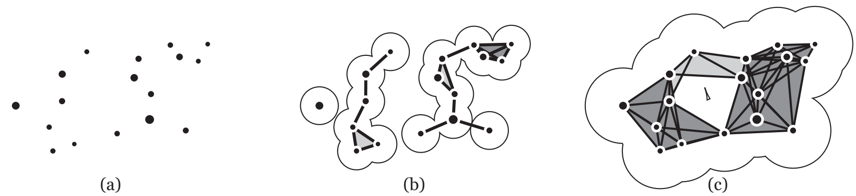


Figure 2.1: The Čech filtration of the flip book example (in the bottom corner of right-side pages), which consists the the points (a) in $\mathbf{R}^2$, at threshold levels $t = 0$, 2.5, and 5 mm, respectively. 0-simplices are represented as points, 1-simplices as edges, 2-simplices as lightly shaded triangles, and 3-simplices as darker quadrilaterals. Higher-dimensional simplices are not shown. The outline is the subset of $\mathbf{R}^2$ whose distance from the points is the threshold level[1].

complex by $\mathrm{VR}_t(X)$:

$$\mathrm{VR}_t(X) \coloneqq \{\sigma \subseteq X \mid \forall x, y \in \sigma,\ d(x,y) \leq t\}.$$

As we can see, these complexes are parameterized by the radius t. This leads us to our construction of filtered simplicial complexes.

Let K be the full simplicial complex on X. The **Čech filtration on** K, denoted by $\check{\mathrm{C}}(X)$, assigns to each simplex $\sigma \in K$ the minimal radius of a closed ball in M that contains σ. The **Vietoris-Rips filtration on** K, denoted by $\mathrm{VR}(X)$, assigns to each simplex $\sigma \in K$ the maximum of the pairwise distances of σ. We denote the sublevel sets of $\check{\mathrm{C}}(X)$ and $\mathrm{VR}(X)$ by $\check{\mathrm{C}}_t(X)$ and $\mathrm{VR}_t(X)$, respectively, and observe that they coincide with the Čech and Vietoris-Rips complexes defined above. Note that the sublevel sets are empty for $t < 0$.

As an example, the flip book in the corner of the right-side pages is constructing the Čech filtration on a set of vertices; see Figure 2.1 for details. ■

We now define simplicial homology over the field of two elements $\mathbf{F}_2$ (for general topological spaces, we may use singular homology [32]), which will give us persistent homology for filtered simplicial complexes. The choice of field is made to simplify the following constructions; these constructions exist over all fields (see, for example, [32]).

2.6 **Definition.** A **chain complex** consists of a sequence of vector spaces $(C_d)_{d\geq 0}$, along with $C_{-1}(K) \coloneqq 0$, and a sequence of linear maps $\partial_d \colon C_d \to C_{d-1}$ such that

[1]The size of each point is purely cosmetic: the vertices are a plot of the brightest stars in the Leo constellation (with some choices made to produce a more interesting topology), and their sizes scale with the brightness of the corresponding stars.

$\partial_{d-1} \circ \partial_d = 0$ for all $d \geq 0$. We call C_d the space of d-**chains** and ∂_d the d^{th} **boundary map**. We denote such a chain complex by (C_*, ∂).

A **morphism of chain complexes** $f\colon (C_*, \partial^C) \to (D_*, \partial^D)$ consists of a sequence of linear maps $f_d\colon C_d \to D_d$ for all $d \geq 0$, along with $f_{-1} := 0$, such that the following square commutes for all $d \geq 0$:

$$\begin{tikzcd} C_d \arrow[r, "\partial_d^C"] \arrow[d, "f_d"'] & C_{d-1} \arrow[d, "f_{d-1}"] \\ D_d \arrow[r, "\partial_d^D"'] & D_{d-1} \end{tikzcd} \tag{2.1}$$

For all $d \geq 0$, elements of $Z_d := \ker \partial_d$ are called d-**cycles**, and elements of $B_d := \operatorname{im} \partial_{d+1}$ are called d-**boundaries**. By definition, we have $\partial_{d-1} \circ \partial_d = 0$ for all $d \geq 0$, so d-boundaries are d-cycles. We therefore define the d^{th} **homology space of** (C_*, ∂) (over $\mathbf{F}_2$) to be the quotient vector space

$$H_d := Z_d / B_d.$$

Note that a morphism of chain complexes induces a morphism on homology: in particular, we check that cycles map to cycles and boundaries to boundaries using (2.1). ■

2.7 **Definition.** Let K be a simplicial complex. For all integers $d \geq 0$, denote by $C_d(K) := \mathbf{F}_2 K_d$ the vector space generated by formal linear combinations of d-simplices. A **simplicial** d-**chain** is an element of $C_d(K)$. We define the **simplicial chain complex of** K as the chain complex $(C_*(K), \partial)$ where the boundary operator ∂ is the $\mathbf{F}_2$-linear map defined on d-simplices by

$$\partial_d(\{v_0, \dots, v_d\}) := \sum_{i=0}^{d} \{v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_d\},$$

and extended to all d-chains by linearity. We can check that $\partial_{d-1}\partial_d = 0$ for all $d \geq 0$. We define the d^{th} **simplicial homology space of** K (over $\mathbf{F}_2$) to be the d^{th} homology $H_d(K)$ of the simplicial chain complex $(C_*(K), \partial)$. ■

We have thus defined simplicial homology for a single simplicial complex; we now turn to the persistent version.

2.8 **Definition.** Let (K, f) be a filtered simplicial complex. For each value $t \in \mathbf{R}$, we consider the simplicial chain complex $C_*(K_{\leq t})$ on the corresponding sublevel set. For all $s \leq t$ in $\mathbf{R}$, the natural inclusions of spaces of simplicial d-chains $C_d(K_{s\leq t})\colon C_d(K_{\leq s}) \hookrightarrow C_d(K_{\leq t})$ commute with the boundary maps, and therefore they extend to a natural inclusion of simplicial chain complexes $C_*(K_{\leq s}) \hookrightarrow C_*(K_{\leq t})$: this data is called a **filtered chain complex**. These inclusions induce the linear maps (not necessarily inclusions) $H_d(K_{s\leq t})\colon H_d(K_{\leq s}) \to H_d(K_{\leq t})$ between homology spaces, and this is the d^{th} **persistent homology of the filtered simplicial complex** (K, f). ■

2.1.2 As the image of a functor

It turns out that persistent homology does not need to come from a filtered topological space. It suffices to have a sequence of maps between spaces

$$X_0 \to X_1 \to \cdots \to X_N$$

and then apply a homology functor. First, we recall some category theory.

2.9 **Definition.** A **category C** is the given of

- a collection $\operatorname{obj}(\mathbf{C})$ of **objects**,
- for every two objects x, y in $\operatorname{obj}(\mathbf{C})$, a collection $\operatorname{mor}_{\mathbf{C}}(x, y)$ of **morphisms**,
- for every three objects x, y, z in $\operatorname{obj}(\mathbf{C})$, a **composition operation**

$$\circ\colon \operatorname{mor}_{\mathbf{C}}(x, y) \times \operatorname{mor}_{\mathbf{C}}(y, z) \to \operatorname{mor}_{\mathbf{C}}(x, z)$$

  that maps the pair (f, g) to a morphisms $g \circ f$, and
- for every object x in $\operatorname{obj}(\mathbf{C})$, an **identity morphism** id_x in $\operatorname{mor}_{\mathbf{C}}(x, x)$,

such that

- composition is **associative**: for every four objects w, x, y, z in $\operatorname{obj}(\mathbf{C})$, f in $\operatorname{mor}_{\mathbf{C}}(w, x)$, g in $\operatorname{mor}_{\mathbf{C}}(x, y)$, and h in $\operatorname{mor}_{\mathbf{C}}(y, z)$,

$$h \circ (g \circ f) = (h \circ g) \circ f),$$

- identity morphisms are **left and right units**: for every two objects x, y in $\operatorname{obj}(\mathbf{C})$ and f in $\operatorname{mor}_{\mathbf{C}}(x, y)$,

$$f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f.$$

In practice, we write $x \in \mathbf{C}$ for an object of $\mathbf{C}$, and $f\colon x \to y$ for a morphism in $\mathrm{mor}_{\mathbf{C}}(x, y)$. ■

2.10 **Example.** Categories encapsulate most common mathematical frameworks: for example, we have the category **Set**, whose objects are sets and morphisms are set-theoretic functions. Another example is the category $\mathbf{Vect}_{\mathbf{k}}$, whose objects are vector spaces over the field $\mathbf{k}$ and whose morphisms are $\mathbf{k}$-linear maps. For persistent homology, the relevant categories are **Top** and **SCpx**, consisting of topological spaces and continuous maps on the one hand, and simplicial complexes and morphisms of simplicial complexes on the other. ■

2.11 **Notation.** The subscript in $\mathrm{mor}_{\mathbf{C}}$ is often omitted when the ambient category is clear. Additionally, for categories of algebraic objects such as groups, rings, and persistence modules, the convention is to write $\hom_{\mathbf{C}}(x, y)$ instead of $\mathrm{mor}_{\mathbf{C}}(x, y)$, and speak of **homomorphisms** (not homeomorphisms!) instead of morphisms. We will adopt the notation but not the terminology in this thesis. ■

2.12 **Definition.** A functor is a way to pass from one category to another. Formally, let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A **functor** $F\colon \mathbf{C} \to \mathbf{D}$ consists of

- for each object x in $\mathbf{C}$, an object $F(x)$ in $\mathbf{D}$, and
- for each pair of objects x, y in $\mathbf{C}$ and morphism $f\colon x \to y$, a morphism $F(f)\colon F(x) \to F(y)$,

such that

- composition is respected: for every three objects x, y, z in $\mathbf{C}$ and morphisms $f\colon x \to y$ and $g\colon y \to z$, $F(g \circ f) = F(g) \circ F(f)$, and
- units are respected: for every object x in $\mathbf{C}$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$.

Let F and G be functors from $\mathbf{C}$ to $\mathbf{D}$. A **natural transformation from** F **to** G consists of a collection $\{\eta_x\}_{c \in \mathbf{C}}$ of morphisms $\eta_x\colon F(x) \to G(x)$ in $\mathbf{D}$ such, for every morphism $f\colon x \to y$ in $\mathbf{C}$, the following square commutes:

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\
G(x) \arrow[r, "G(f)"'] & G(y)
\end{tikzcd}$$

We denote such a natural transformation by $\eta\colon F \Rightarrow G$. Natural transformations serve as morphisms between functors, and thus we naturally obtain a **category of functors from C to D** denoted by $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$.

Sometimes two categories behave very similarly. For example, we will see several definitions of persistence modules, and some of these will be equivalent. We formalize this notion as follows. An **equivalence of categories** between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of

- a pair of functors $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{C}$, and
- **natural isomorphisms** (i.e., invertible natural transformations) $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{C}}$.

Two categories are **equivalent** if there is an equivalence of categories between them. ■

2.13 **Example.** Fix a field $\mathbf{k}$. An example of a functor is the one from **Set** to $\mathbf{Vect_k}$ that maps a set X to the vector space $\mathbf{k}X$ generated by formal sums of elements of X, and maps a set-valued function $f\colon X \to Y$ to the linear map that sends each basis element x in $\mathbf{k}X$ to the corresponding basis element $f(x)$ in $\mathbf{k}Y$. ■

It can be shown that homology is a functor from **Top** to $\mathbf{Vect_k}$ [32]. For example, the construction of the homology spaces in Definition 2.7 is functorial. We can thus view persistent homology more generally as the application of the homology functor to a subcategory of topological spaces (or simplicial complexes). Specifically, let I be a totally ordered set, $(X_t)_{t\in I}$ a collection of spaces (or simplicial complexes), and $X_{s\le t}\colon X_s \to X_t$ a morphism for each $s \le t$ in I, satisfying the composition and identity axioms for categories (see Definition 2.9). This defines a category, which we denote by X, whose objects are the vector spaces X_t and morphisms are the linear maps $X_{s\le t}$. For all integers $d \ge 0$, we then call the image of X by the functor $H_d(-)$ the d^{th} **persistent homology of** X.

2.1.3 As a collection of vector spaces and transition maps

The image of the aforementioned homology functors is interesting in its own right. Abstracting the previous discussion, we can view persistent homology as a collection of vector spaces and linear maps between them. This is how we define persistence modules:

2.14 **Definition.** Fix a field $\mathbf{k}$ and $(I, \leq)$ a totally ordered set. A **persistence module over** I is a collection of $\mathbf{k}$-vector spaces $(V_t)_{t \in I}$ indexed by I, along with linear maps $(V_{s \leq t} \colon V_s \to V_t)_{s \leq t \in I}$, called **transition maps**, such that

- for all $r \leq s \leq t$ in $\mathbf{R}$, we have the equality $V_{s \leq t} \circ V_{r \leq s} = V_{r \leq t}$, and
- for all t in I, $V_{t \leq t} = \mathrm{id}_{V_t}$.

Let V and W be persistence modules. A **morphism from** V **to** W is a collection of linear maps $(f_t \colon V_t \to W_t)_{t \in I}$ such that, for all $s \leq t$ in I, the following diagram commutes:

$$\begin{tikzcd} V_s \arrow[r, "V_{s \leq t}"] \arrow[d, "f_s"'] & V_t \arrow[d, "f_t"] \\ W_s \arrow[r, "W_{s \leq t}"'] & W_t \end{tikzcd} \tag{2.2}$$

■

The similarities between these conditions and those of functors (Definition 2.12) is not a coincidence, and will be detailed further in Section 2.1.5.

2.1.4 As a graded module

Persistence modules have been extensively studied as graded modules over polynomial rings [19, 38, 52]. To simplify things, we present here the case where the indexing poset is $\mathbf{N}$. We have the following equivalent characterization of persistence modules:

2.15 **Proposition.** *Let* $\mathbf{k}$ *be a field. A persistence module over* $\mathbf{N}$ *is equivalently a graded module over the polynomial ring* $\mathbf{k}[x]$.

Proof. Let V be a persistence module. We construct the graded module $\bigoplus_{t=0}^{\infty} V_t$ where each V_t has grade t, and the action of $\mathbf{k}[x]$ is determined by x acting on V_t as $V_{t \leq t+1}$.

Conversely, let V be a graded module over $\mathbf{k}[x]$. We define V_t to be the summand of V of grade t, and, for all $s \leq t$, we define $V_{s \leq t}$ to be the action of x^{t-s} on V_s. We check that this satisfies the associativity and unit conditions of Definition 2.14. □

We remark that nothing prevents us from considering multigraded modules over multivariate polynomial rings. This is the approach of [15], where the authors call such modules multidimensional persistence. Nowadays, they are more commonly called **multiparameter persistence modules**, and we immediately generalize this further to arbitrary posets.

2.1.5 As a functor itself

In Section 2.1.3, we noted that the definition of a persistence module closely resembles that of a functor. In Section 2.1.4, we remarked that persistence modules may be indexed by more general posets than totally ordered sets. We formalize both of these observations here:

2.16 **Definition.** Let $\mathbf{k}$ be a field and $(I, \leq)$ a poset. Note that $(I, \leq)$ is a category (Definition 2.9) where $\mathrm{obj}(I, \leq) = I$ and, for all s, t in I, $\mathrm{mor}_{(I,\leq)}(s,t) = \{*\}$ if $s \leq t$ and $\varnothing$ otherwise. We define a **generalized persistence module indexed by** (or **over**) I to be a functor $V\colon (I, \leq) \to \mathbf{Vect}_{\mathbf{k}}$.

A **morphism of persistence modules** is then a natural transformation between persistence modules. ∎

We check that this definition coincides with Definition 2.14 when I is totally ordered. **From now on, the term "persistence module" refers to this functorial definition.**

2.1.6 As a sheaf

Pushing the generalization of persistence modules even further, we can view them through the lens of sheaf theory [6, 35]. Here we present a brief overview of what this theory entails. We start with some preliminary definitions in category theory.

2.17 **Definition.** Let $\mathbf{C}$ be a category. The **opposite category** of $\mathbf{C}$ is the category $\mathbf{C}^{\mathrm{op}}$ where $\mathrm{obj}(\mathbf{C}^{\mathrm{op}}) := \mathrm{obj}(\mathbf{C})$ and, for all x, y in $\mathrm{obj}(\mathbf{C}^{\mathrm{op}})$, $\mathrm{mor}_{\mathbf{C}^{\mathrm{op}}}(x,y) := \mathrm{mor}_{\mathbf{C}}(y,x)$. Composition is that of $\mathbf{C}$, but inverted: writing $f^{\mathrm{op}}\colon y \to x$ and $g^{\mathrm{op}}\colon z \to y$ as the elements in $\mathbf{C}^{\mathrm{op}}$ that correspond to $f\colon x \to y$ and $g\colon y \to z$ in $\mathbf{C}$, we define $f \circ^{\mathrm{op}} g := g \circ f$. ∎

For example, the opposite category of a poset $(I, \leq)$, viewed as a category, is the opposite poset $(I, \geq)$.

2.18 **Definition.** Let X be a topological space and denote by $\mathscr{O}(X)$ the poset of open sets of X. A **Vect-valued presheaf on** X is a functor $F\colon \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Vect}$. For U an open set in $\mathscr{O}(X)$, the elements of $F(U)$ are called **sections of** F **over** U. For each containment of open sets $V \supseteq U$ in $\mathscr{O}(X)$, we get a **restriction morphism** $(-)|_U\colon F(V) \to F(U)$. The presheaf F is then a **sheaf** if it satisfies the following two axioms:

1. **Locality:** for U an open set, $\{U_\alpha\}_{\alpha \in A}$ an open cover of U with $U_\alpha \subseteq U$ for all $\alpha \in A$, and $v, w \in F(U)$, if $v|_{U_\alpha} = w|_{U_\alpha}$ for all $\alpha \in A$, then $v = w$. In other words, if two sections coincide locally, then they are equal globally.
2. **Gluing:** for U an open set, $\{U_\alpha\}_{\alpha \in A}$ an open cover of U with $U_\alpha \subseteq U$ for all $\alpha \in A$, and $v_\alpha \in F(U_\alpha)$ for all $\alpha \in A$, if $v_\alpha|_{U_\alpha \cap U_\beta} = v_\beta|_{U_\alpha \cap U_\beta}$ for all α, β in A, then there exists v in $F(U)$ such that $v|_{U_\alpha} = v_\alpha$ for all $\alpha \in A$. In other words, if local sections coincide on their overlapping parts, then they can be glued together into a global section.

Now let us equip our poset $(I, \leq)$ with a topology. The **Alexandrov topology** on I is the topology where open sets are **upsets**, that is, subsets S of I such that $S = \{t \in I \mid \exists s \in S,\ s \leq t\}$. We check that this is indeed a topology. ∎

2.19 **Proposition** ([6, Proposition 2.2.3], [35, Proposition 2.15(b)]). *Let $(I, \leq)$ be a poset, equipped with the Alexandrov topology. Then persistence modules indexed by I are equivalent (in the categorical sense, see Definition 2.12) to* **Vect**-*valued sheaves over I.*

2.1.7 As a barcode

We have strayed far enough our persistent homology pipeline. Let us return to the **one-dimensional** case, that is, the case of generalized persistence modules where the indexing poset I is a total order. The remarkable result that gives persistent homology its expressive power is the fact that one-dimensional persistence modules admit a decomposition theorem.

2.20 **Definition.** Let $(I, \leq)$ be a totally ordered set. An **interval** of I is a subset S such that, for all, s, t in S and r in I such that $s \leq r \leq t$, r is also an element of S.

Suppose that $(I, \leq)$ is a total order and $S \subseteq I$ an interval. The **interval module** or **bar with support** S is the persistence module $\mathbf{k}_S$ defined functorially by, for all $s \leq t$ in I,

$$\mathbf{k}_S(t) := \begin{cases} \mathbf{k} & \text{if } t \in S, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathbf{k}_S(s \leq t) := \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } s, t \in S, \\ 0 & \text{otherwise.} \end{cases}$$ ■

2.21 **Definition.** A **pointwise finite-dimensional (p.f.d.) persistence module** is a persistence module V indexed by a poset $(I, \leq)$ such that, for all $t \in I$, $V(t)$ is a finite-dimensional vector space. ■

2.22 **Definition.** A **multiset** is a set X equipped with a **multiplicity** function $m_X : X \to \mathbf{N} \setminus \{0\}$. A multiset is **finite** if its underlying set is finite. We often omit the multiplicity function when writing multisets.

We can iterate over a multiset X by indexing its elements by a set. For instance, we can write $X = \{x_\alpha\}_{\alpha \in A}$ with A the indexing set. ■

2.23 **Theorem** ([9, Theorem 1.2]). *Let V be a pointwise finite-dimensional persistence module indexed by a total order $(I, \leq)$. Then there exists a unique multiset of intervals $\{S_\alpha\}_{\alpha \in A}$ of I such that*

$$V \cong \bigoplus_{\alpha \in A} \mathbf{k}_{S_\alpha}.$$

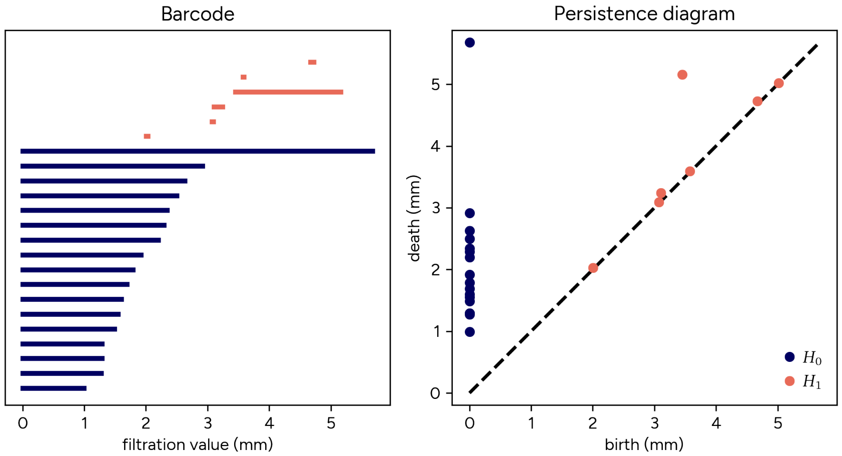
2.24 **Definition.** We call this multiset of intervals the **barcode of** V, and denote it by $\mathscr{B}(V)$. We call the isomorphism (and the direct sum of bars) a **barcode decomposition of** V.

Suppose that I is the real line[2] and let S be an interval. We define the **birth** or **start-point** of S to be $s(S) := \inf S$ and the **death** or **end-point** of S to be $t(S) := \sup S$. ■

Note that the collection of intervals may be infinite and even uncountable: this is the case when $I = \mathbf{R}$. This is somewhat unwieldy, and oftentimes results are shown for more finite (and numerically manageable) persistence modules.

2.25 **Definition.** A **tame persistence module** is a pointwise finite-dimensional persistence module V over a total order $(I, \leq)$ such that there exists a finite

[2]A **bounded complete** poset suffices, that is, a poset where every subset with an upper bound admits a least upper bound.



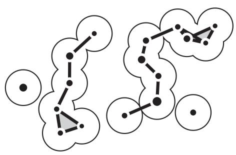
(a) (b)

Figure 2.2: Barcodes (a) and persistence diagrams (b) in degrees 0 and 1 for the flip book example (see Figure 2.1).

sequence $t_1 < \cdots < t_n$ such that, for $s \leq t$ in I, $V(s \leq t)$ is a non-isomorphism only if $s < t_k \leq t$ for some $k \in \{1, \ldots, n\}$. In this case, the barcode decomposition must consist of tame interval modules: each interval is of the form $[s, t)$, that is, closed from below and open from above [52]. ■

2.26 **Example.** For example, in the case where I is the real line and V is the persistent homology of a finite filtered simplicial complex, V is tame. The bars of its decomposition would then correspond to a topological feature (depending on the degree of homology: a connected component, a loop, a cavity, etc.) that appears at filtration level s and disappears at filtration level t. The further these values are from each other, the longer the topological feature *persists*: hence the name of persistent homology. See Figure 2.2a for an example of the barcodes in degrees 0 and 1. ■

2.27 **Remark.** Barcode decompositions do not exist for more general persistence modules. This statement was formalized in [15], which proved that there is no complete, discrete invariant for multiparameter persistence modules. Instead, the authors propose to look at the rank invariant, a discrete invariant for arbitrary indexing sets: see Section 2.1.10 for more details. ■



2.1.8 As a diagram

Let V be a (pointwise finite-dimensional) persistence module indexed by $\mathbf{R}$. We can represent its barcode as a multiset of points in $\mathbf{R}^2$.

2.28 **Definition.** Let V be a persistence module indexed by $\mathbf{R}$. Recall from Theorem 2.23 that its barcode is a multiset of intervals $\{S_\alpha\}_{\alpha \in A}$. Recall that to each interval S_α we associate a start-point $s(S_\alpha)$ and end-point $t(S_\alpha)$. The **persistence diagram** of V is the multiset of points $\mathscr{D}(V) := \{(s(S_\alpha), t(S_\alpha))\}_{\alpha \in A}$ in $\mathbf{R}^2$. These points are all above the line $y = x$.

Persistence diagrams can also be viewed as discrete measures on $\mathbf{R}^2$ [17]. Specifically, the **persistence measure** of V is the sum of Dirac measures

$$\mu_V := \sum_{\alpha \in A} \delta_{(s(S_\alpha), t(S_\alpha))}. \qquad \blacksquare$$

2.29 **Example.** Figure 2.2b is the persistence diagram in degrees 0 and 1 of the running Čech filtration example. ■

2.30 **Remark.** Persistence diagrams contain less information than barcodes. Specifically, they forget about the inclusion of start- and end-points in the intervals. For tame persistence modules, this is not an issue, as all intervals are of the same form (closed below and open above), but this is relevant for p.f.d. modules. ■

2.1.9 As a point in a metric space

In TDA, not only are we interested in the topological features of a data set, but we would also like to compare the topologies of different data sets. To do this, we define (pseudo)metrics on the space of persistence diagrams.

2.31 **Definition.** Let X and Y be sets. A **(partial) matching of** X **and** Y is a subset M of $X \times Y$ such that each element of X and each element of Y appears at most once. We denote by $\mathbf{Match}(X, Y)$ the set of all matchings between X and Y.

A partial matching of multisets is defined similarly, as a sub-multiset M of $X \times Y$ (where the multiplicity of (x, y) is $m_{X\times Y}(x, y) := m_X(x)m_Y(y)$), such that each element x of X (resp. y of Y) appears at most $m_X(x)$ (resp. $m_Y(y)$) times.

Let D and D' be persistence diagrams. Denote by Δ the diagonal $\{(t, t) \mid t \in \mathbf{R}\}$ in $\mathbf{R}^2$. Fix $p \in [1, +\infty]$ and denote by d_p the p-norm distance in $\mathbf{R}^2$: recall that, for $x, y \in \mathbf{R}^2$,

$$d_p(x,y) := \|x-y\|_p := \begin{cases} \left((x_1-y_1)^p + (x_2-y_2)^p\right)^{\frac{1}{p}} & \text{if } p<\infty, \\ \max(|x_1-y_1|, |x_2-y_2|) & \text{if } p=+\infty. \end{cases}$$

Let M be a matching of D and D'. We define the **p-cost of M** to be the number

$$c_p(M) := \left(\sum_{(a,b)\in M} d_p(a,b)^p + \sum_{\substack{a\in D\\ \text{unmatched}}} d_p(a,\Delta)^p + \sum_{\substack{b\in D'\\ \text{unmatched}}} d_p(b,\Delta)^p \right)^{\frac{1}{p}}$$

if $p<\infty$ and

$$\max\left(\sup_{(a,b)\in M} d_\infty(a,b), \sup_{\substack{a\in D\\ \text{unmatched}}} d_\infty(a,\Delta), \sup_{\substack{b\in D'\\ \text{unmatched}}} d_\infty(b,\Delta) \right)$$

if $p=\infty$. That is, $c_p(M)$ is the p-norm of the vector of distances between matched pairs of points and distances between unmatched points and the diagonal.

We define the **p-Wasserstein distance** between the persistence diagrams D and D' to be

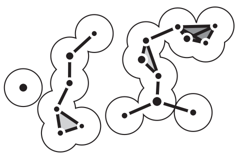
$$d_p(D,D') := \inf_{M\in\mathbf{Match}(D,D')} c_p(M).$$

When $p=+\infty$, we get the ∞-Wasserstein distance, which is more commonly referred to as the **bottleneck distance**.

These distances between persistence diagrams lift to distances between barcodes: given persistence modules V and W and $p\in[1,+\infty]$, we define

$$d_p(\mathscr{B}(V),\mathscr{B}(W)) := d_p(\mathscr{D}(V),\mathscr{D}(W)).$$ ■

2.32 **Remark.** The distance functions here are more formally defined as **pseudometrics**: they satisfy the axioms of symmetry and triangular inequality for metrics [24], but they do not necessarily separate points: for instance, a persistence diagram whose points are all on the diagonal would be distance 0 from the empty diagram. If we exclude the points on the diagonal, which is sometimes done in the literature [16], then these distances become metrics in the formal sense. ■



2.33 **Remark**[3]. The name "Wasserstein distance" is due to [24], where the infimum in the definition iterates over all *total* matchings (i.e., every point is matched) of persistence diagrams unioned with the points on the diagonal with infinite multiplicity. This is equivalent to our definition, but is more accurate to the Wasserstein, or earth-mover's, distance in optimal transport theory for measures, where matchings are total in a measure-theoretic sense. The "totality" of the matchings in [24] is rather artificial, as we include the diagonal infinitely many times. We will prefer our definition with partial matchings, but then the more appropriate name might be Figalli-Gigli distance, named after [26]. See [22] and [34, Remark 2] for further discussion about this point. ■

A fundamental property of persistent homology is its **stability**: small perturbations of the input filtered space result in small changes in the output persistence diagrams. This is usually formalized as a Lipschitz-type inequality:

2.34 **Theorem** ([24], [46, Theorem 4.8]). *Let X be a topological space or a simplicial complex, $f, g\colon X \to \mathbf{R}$ filtrations on X, and V and W the associated persistence diagrams. Then, for all $p \in [1, \infty]$, we have the inequality*

$$d_p(\mathscr{D}(V), \mathscr{D}(W)) \leq \|f - g\|_p.$$

We can also define distances on persistence modules.

2.35 **Definition.** Let V and W be persistence modules over $\mathbf{R}$ and $\varepsilon > 0$ a real number. The **ε-shift** of V is the persistence module $V[\varepsilon]$, where, for all $s \leq t$ in $\mathbf{R}$,

$$V[\varepsilon](t) := V(t+\varepsilon) \quad \text{and} \quad V[\varepsilon](s \leq t) := V(s+\varepsilon \leq t+\varepsilon).$$

Given a morphism of persistence modules $f\colon V \to W$, the **ε-shift** of f is the morphism $f[\varepsilon]$ where, for all t in $\mathbf{R}$, $f[\varepsilon]_t := f_{t+\varepsilon}$.

We denote by $V(\cdot \leq \cdot + \varepsilon)\colon V \to V[\varepsilon]$ the morphism defined, for all t in $\mathbf{R}$, by $V(\cdot \leq \cdot + \varepsilon)_t := V(t \leq t+\varepsilon)\colon V(t) \to V[\varepsilon](t) = V(t+\varepsilon)$.

An **ε-interleaving between V and W** is a pair of morphisms $\varphi\colon V \to W[\varepsilon]$ and $\psi\colon W \to V[\varepsilon]$ such that

$$\psi[\varepsilon] \circ \varphi = V(\cdot \leq \cdot + 2\varepsilon) \quad \text{and} \quad \varphi[\varepsilon] \circ \psi = W(\cdot \leq \cdot + 2\varepsilon).$$

[3]Thank you to Théo Lacombe for bringing this to my attention.

Diagrammatically, we have the following two commuting triangles:

$$\begin{tikzcd}
V(t) \arrow[rr, "V(t\leq t+\varepsilon)"] \arrow[drr, "\varphi_t" near start] & & V(t+\varepsilon) \arrow[rr, "V(t+\varepsilon\leq t+2\varepsilon)"] \arrow[drr, "\varphi_{t+\varepsilon}" near start] & & V(t+2\varepsilon) \\
W(t) \arrow[rr, "W(t\leq t+\varepsilon)"'] \arrow[urr, "\psi_t" near start] & & W(t+\varepsilon) \arrow[rr, "W(t+\varepsilon\leq t+2\varepsilon)"'] \arrow[urr, "\psi_{t+\varepsilon}" near start] & & W(t+2\varepsilon)
\end{tikzcd}$$

The **interleaving distance between** V **and** W is then

$$d_I(V,W) \coloneqq \inf\{\varepsilon \mid \exists\varepsilon\text{-interleaving between } V \text{ and } W\}. \qquad \blacksquare$$

2.36 **Remark.** As with the distances defined between persistence diagrams and barcodes, the interleaving distance is not formally a metric. In particular, it does not separate isomorphic persistence modules, which can be shown to admit 0-interleavings. We will consider distances d between persistence modules as **pseudometrics** satisfying the following axioms: for all persistence modules U, V, W,

1. $d(U,V) = d(V,U)$;
2. $d(U,V) = 0$ if U is isomorphic to V;
3. $d(U,W) \leq d(U,V) + d(V,W)$. ■

Remarkably, the interleaving distance on persistence modules and the bottleneck distance on the corresponding persistence diagrams and barcodes are equal:

2.37 **Theorem** ([3, Theorem 3.5]). *Let V and W be two (pointwise finite-dimensional) persistence modules over $\mathbf{R}_{\leq}$. Then*

$$d_I(V,W) = d_\infty(\mathscr{B}(V), \mathscr{B}(W)).$$

One of the results of this thesis is the proof of similar results for other p-Wasserstein distances. To do this, we need to define an algebraic p-Wasserstein distance for persistence modules: this is the content of Papers B and C, which provide an alternative proof that the construction of [46] is indeed a pseudometric.

2.1.10 As its rank function

As noted in Remark 2.27, generalized persistence modules do not have barcodes. However, if we allow ourselves to forget some information, then we can instead study the rank function of a persistence module instead.

2.38 **Definition.** Let V be a persistence module indexed by $(I,\leq)$. The **rank function of** V is the function

$$\operatorname{rank} V\colon \begin{cases} \Omega(I) & \to & \mathbf{N} \\ (s \leq t) & \mapsto & \operatorname{rank} V(s \leq t), \end{cases}$$

where $\Omega(I)$ is the set $\{(s,t) \in I \times I \mid s \leq t\}$. ■

In the one-dimensional case, the rank function is equivalent to the barcode in the sense that either one can be recovered from the other [15]. In the general case, rank functions decompose as the signed sum of rank functions of simple modules, such as rectangles [10].

2.2 Persistent linear algebra

Having defined persistence modules, we now turn to their study through the lens of linear algebra. Recall that a persistence module is a functor from a poset $(I,\leq)$ to **Vect**. In particular, this means that many notions that are defined for vector spaces also apply to persistence modules. We begin with some preliminary notions:

2.39 **Definition.** Direct sums, kernels, and cokernels of persistence modules all exist and are defined pointwise[4]. That is, given persistence modules V and W, a morphism $f\colon V \to W$, and $t \in I$, we have

- $(V \oplus W)(t) = V(t) \oplus W(t)$,
- $(\ker f)(t) = \ker(f_t\colon V(t) \to W(t))$,
- $(\operatorname{coker} f)(t) = \operatorname{coker}(f_t\colon V(t) \to W(t))$.

There are equivalents of injective and surjective maps: a morphism f is

- a **monomorphism** if every f_t is injective, and

[4]This is true in general for functor categories (Definition 2.12) with values in **Vect** or any other abelian category.

- an **epimorphism** if every f_t is surjective. ∎

2.2.1 Persistence vectors and generators

Considering persistence modules that are graded modules (Section 2.1.4), there is an immediate notion of an "element" of a persistence module, which we will call a **persistence vector**. We first define this for tame persistence modules over $\mathbf{R}$. This section is in part inspired by the framework of [50]; compare also with [13, Section 6]. First, we define the notion of persistence vectors.

2.40 **Definition.** Let V be a tame persistence module over $\mathbf{R}$. A **persistence vector of** V is an element v in $V(s)$, for some $s \in \mathbf{R}$. Equivalently, it is the given of a map $v\colon \mathbf{k}_{[s,+\infty)} \to V$, since such a map is determined by its value at s: for all $t \geq s$, we have $v_t = V(s \leq t) \circ v_s$, and for all $t \ngeq s$, we have $v_t = 0$. We call s the **grade** of v, and write $\mathrm{grade}(v) := s$. Viewing v as a function, we denote by $\mathrm{death}(v) := \inf\{t \geq I \mid v(t) = 0\}$ the **death time** of v. ∎

Next, we define persistent versions of generating sets and linear independence.

2.41 **Definition.** Let $S := \{v_\alpha\}_{\alpha \in A}$ be a set of persistence vectors of V and denote by s_α the grade of v_α for all $\alpha \in A$. We say that S is a **generating set of** V if, for every (standard) vector w in the vector space $V(s)$, there exists a subset $A' \subseteq A$ and coefficients $\lambda_\alpha \in \mathbf{k}$ for all $\alpha \in A'$ such that

$$w = \sum_{\alpha \in A'} \lambda_\alpha V(s_\alpha \leq s)(v_\alpha). \tag{2.3}$$

In particular, each α in A' must satisfy the inequality $s_\alpha \leq s$.

We say that S is **linearly independent** if, for all $t \in \mathbf{R}$, the set $\{v_\alpha(t) \mid \alpha \in A, v_\alpha(t) \neq 0\}$ is linearly independent as standard vectors in $V(t)$. ∎

Recall from Definition 2.25 that we have a barcode decomposition of V, which is an isomorphism, of the form

$$\varphi\colon \bigoplus_{\alpha \in A} \mathbf{k}_{[s_\alpha, t_\alpha)} \to V.$$

Note that φ decomposes into a sum of morphisms $\varphi_\alpha\colon \mathbf{k}_{[s_\alpha,t_\alpha)} \to V$. Each morphism φ_α necessarily lifts to a morphism $\varphi'_\alpha\colon \mathbf{k}_{[s_\alpha,+\infty)} \to V$, where, for all $t \geq b_\alpha$, $(\varphi'_\alpha)_t$ is determined by the value of φ_α at s_α. Thus φ can be seen as the sum of persistence vectors in their functional definition. Moreover, by construction, these persistence vectors φ'_α generate V. This motivates the following definition.

2.42 **Definition.** Let V be a tame persistence module over $\mathbf{R}$ and $B := \{v_\alpha : \mathbf{k}_{[s_\alpha,+\infty)} \to V\}_{\alpha\in A}$ a collection of persistence vectors of V. For each $\alpha \in A$, denote by t_α the death time of v_α. Then v_α factors through $\varphi_\alpha : \mathbf{k}_{[s_\alpha,t_\alpha)} \to V$.

The collection B is a **persistence basis of** V if the morphism

$$\sum_{\alpha\in A} \varphi_\alpha : \bigoplus_{\alpha\in A} \mathbf{k}_{[s_\alpha,t_\alpha)} \to V$$

is an isomorphism. We also refer to this isomorphism as a persistence basis. ■

2.43 **Remark.** The equivalent notion in [50] is called a *barcode basis*. In addition, a collection of persistence vectors is a persistence basis if, and only if, it generates V and is linearly independent [50, Proposition 3.11]. ■

Persistence vectors can be generalized to persistence modules over arbitrary posets and to p.f.d. (pointwise finite-dimensional) persistence modules over $\mathbf{R}$, but these two generalizations to do not necessarily coincide.

2.44 **Definition.** Fix $(I, \leq)$ a poset and s an element of I. The **free module at** s is the persistence module $\mathbf{k}_{\{I\geq t\}}$, where $\{I \geq s\} := \{t \in I \mid t \geq s\}$ is an upset of I (Definition 2.18).

Let V be a persistence module over I. A **persistence vector of** V is an element v in $V(s)$. As before, we call s the grade of v, and write $\mathrm{grade}(v) := s$. Unlike before, there is no generalized notion of death time (or not a unique one, at least). The notion of **generating sets** is the same as that of (2.3).

Now let V be a p.f.d. persistence module over $\mathbf{R}$. A **persistence vector of** V is a map $v: \mathbf{k}_U \to V$, where U is an upset (Definition 2.18). Explicitly, U is an interval of the form $[s,\infty)$ or (s,∞). We denote by $\mathscr{L}(v) := \{t \in \mathbf{R} \mid v(t) \neq 0\}$ the **support of** v. By the properties of persistence modules, $\mathscr{L}(v)$ is an interval with either closed or open ends.

For p.f.d. modules over $\mathbf{R}$, the notions of grade and death time can defined as the start- and end-point of the support of the persistence vector, but these definitions are not so useful, as they do not encode the closed or opened nature of the support $\mathscr{L}(v)$. The notion of **generating sets** is almost that of (2.3), except that we use the functional notation:

$$w = \sum_{\alpha \in A'} \lambda_\alpha V_{s_\alpha \leq s} \circ v_\alpha.$$

Moreover, we recover the notion of **persistence bases** from the barcode decomposition isomorphism

$$\varphi \colon \bigoplus_{\alpha \in A} \mathbf{k}_{S_\alpha} \to V$$

of Theorem 2.23: we write φ as a sum of morphisms $\varphi_\alpha \colon \mathbf{k}_{S_\alpha} \to V$, each of which lifts to a persistence vector $v_\alpha \colon K_{U_\alpha} \to V$, with $U_\alpha := \{t \in \mathbf{R} \mid \exists s \in S_\alpha,\ t \geq s\}$ the upset generated by S_α. These persistence vectors form a persistence basis of V. ■

2.2.2 Matrix representations

Just as there are many ways of looking at persistence modules, there are also many ways of looking at morphisms of persistence modules. In this section, we consider the matrix perspective for morphisms of tame persistence modules over $\mathbf{R}$. The first idea behind representing morphisms of persistence modules as matrices comes from the isomorphism

$$\hom\Big(\bigoplus_{i \in I} \mathbf{k}_{S_i}, \bigoplus_{j \in J} \mathbf{k}_{T_j}\Big) \cong \bigoplus_{\substack{i \in I \\ j \in J}} \hom(\mathbf{k}_{S_i}, \mathbf{k}_{T_j}).$$

2.45 **Remark.** A similar isomorphism holds for p.f.d. persistence modules over $\mathbf{R}$, except that the righthand side is $\prod_i \bigoplus_j \hom(\mathbf{k}_{S_i}, \mathbf{k}_{T_j})$ instead of the above direct sum. In fact, the isomorphism with the product holds for all abelian categories by the universal properties of products and coproducts. The direct sum holds for tame modules only because the product is finite.

As an example where the product is not direct sum, consider the morphism of p.f.d. modules $f \colon \mathbf{k}_{(0,1)} \to \bigoplus_{n \geq 1} \mathbf{k}_{(0,\frac{1}{n})}$ where, for every $t \in (0,1)$, f_t maps $1_{\mathbf{k}}$ to the vector of all 1's in $\mathbf{k}_{(0,\frac{1}{n})}(t)$. Matrix representations for p.f.d. modules are studied in Paper C. ■

2.46 **Definition** ([4]). Let S and T be intervals of $\mathbf{R}$. We say that T **overlaps** S **above** (and S **overlaps** T **below**) if $S \cap T \neq \varnothing$, every element $s \in S$ satisfies the inequality $s \leq t$ for some $t \in T$, and every element $t \in T$ satisfies the inequality $s \leq t$ for some $s \in S$. ■

The second idea behind matrix representations is the isomorphism, for intervals S and T of $\mathbf{R}$,

$$\hom(\mathbf{k}_S, \mathbf{k}_T) \cong \begin{cases} \mathbf{k} & \text{if } T \text{ overlaps } S \text{ above,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

When T overlaps S above, the morphisms of $\hom(\mathbf{k}_S, \mathbf{k}_T)$ are all of the form λf_0, where $\lambda \in \mathbf{k}$ and $f_0 \colon \mathbf{k}_S \to \mathbf{k}_T$ has identity components over $S \cap T$. We choose the isomorphism to $\mathbf{k}$ that maps f_0 to $1_{\mathbf{k}}$. We now define our matrix representations.

2.47 **Definition.** Let $f \colon V \to W$ be a morphism in **Tame** with fixed persistence bases $\varphi \colon \bigoplus_{j \in J} \mathbf{k}_{[a_j, b_j)} \cong V$ and $\psi \colon \bigoplus_{i \in I} \mathbf{k}_{[c_i, d_i)} \cong W$ (note the inverted indices). For all i in I and j in J, denote by f_{ij} the restriction of the morphism

$$\psi f \varphi^{-1} \colon \bigoplus_{j \in J} \mathbf{k}_{[a_j, b_j)} \to \bigoplus_{i \in I} \mathbf{k}_{[c_i, d_i)}$$

to $\mathbf{k}_{[a_j, b_j)}$ and $\mathbf{k}_{[c_i, d_i)}$: explicitly, $f_{ij} := (\psi f \varphi^{-1})|_{\mathbf{k}_{[a_j, b_j)}}$.

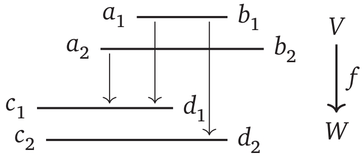
A **matrix representation of** f is a labeled matrix $F = (F_{ij})_{i,j}$ in $\mathbf{k}^{I \times J}$ where $F_{ij} \in \mathbf{k}$ is the identification of f_{ij} via (2.4). Row i is labeled by the i^{th} element of the persistence basis of W, which we denote by its support $[c_i, d_j)$. Similarly, column j is labeled by the j^{th} element of the persistence basis of V, which we denote by its support $[a_j, b_j)$.

A matrix representation is **ordered** if the intervals $\{[a_j, b_j)\}_j$ and $\{[c_i, d_i)\}_j$ are ordered in a way that is compatible with the product order on start- and end-points[5]. In particular, for all j, j' in J, if $[a_{j'}, b_{j'})$ overlaps $[a_j, b_j)$ above, then $j' > j$.

Matrix multiplication is not exactly the same as in linear algebra. Let $f \colon U \to V$ and $g \colon V \to W$ be morphisms of tame persistence modules with fixed persistence bases $\bigoplus_k \mathbf{k}_{[y_k, z_k)} \cong U$, $\bigoplus_j \mathbf{k}_{[a_j, b_j)} \cong V$, and $\bigoplus_i \mathbf{k}_{[c_i, d_i)} \cong W$, and F and G matrix representations of f and g, respectively. The **overlap multiplication** GF is given by

$$(GF)_{ik} = \sum_j G_{ij} F_{jk} \mathbf{I}\{a_j \leq y_k < b_j \leq z_k\},$$

[5]The **product order** $\leq_{\text{prod}}$ is given by $(a_j, b_j) \leq_{\text{prod}} (c_i, d_i)$ if $a_j \leq c_i$ and $b_j \leq d_i$.



$$F = \begin{array}{c} \\ (c_1,d_1) \\ (c_2,d_2) \end{array} \begin{array}{c} \begin{array}{cc} (a_2,b_2) & (a_1,b_1) \end{array} \\ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Figure 2.3: Representation of a morphism $f\colon K(a_1,b_1) \oplus K(a_2,b_2) \to K(c_1,d_1) \oplus K(c_2,d_2)$ and its only ordered matrix representation F (with coefficients in $\mathbf{F}_2$). An arrow between two bars $K(a_j,b_j)$ and $K(c_i,d_i)$ corresponds to a nonzero coefficient F_{ij}.

where $\mathbf{I}\{a_j \leq y_k < b_j \leq z_k\} = 1$ if the inequalities hold and 0 otherwise. We show in Proposition C.16 that this corresponds to a matrix representation of $g \circ f$. ■

2.48 **Example.** See the right-hand side of Figure 2.3 for an example of an ordered matrix representation. In this case, the morphism $f\colon K(a_1,b_1) \oplus K(a_2,b_2) \to K(c_1,d_1) \oplus K(c_2,d_2)$ only has one ordered matrix representation. ■

2.49 **Remark.** This matrix theory works to some extent for persistence modules over arbitrary posets when the morphism $f\colon V \to W$ is between direct sums of modules. For instance, a **free persistence module** is a sum $\bigoplus_{s\in I} \mathbf{k}_{\{I\geq s\}}^{\oplus \beta_s}$ of free modules at poset elements, where β_s is the multiplicity of $\mathbf{k}_{\{I\geq s\}}$ in the sum. We can do a similar analysis to (2.4) on the hom-space $\hom(\mathbf{k}_{\{I\geq s\}}, \mathbf{k}_{\{I\geq t\}})$, and then the matrix representation of f would have rows and columns labeled by the summands of V and W. This is useful when reasoning about projective resolutions: see Paper A, and in particular the implementation [44]. ■

2.50 **Notation.** Extending the representation of barcodes in Figure 2.2, we represent morphisms between barcode decompositions schematically by arrows between bars. Given two barcode decompositions $V := \bigoplus_{\alpha\in A} \mathbf{k}_{S_\alpha}$ and $W := \bigoplus_{\beta\in B} \mathbf{k}_{T_\beta}$ and a morphism $f\colon V \to W$, we place a representation of V above one of W, on the same horizontal scale, and then draw an arrow from the segment S_α to the segment T_β if, and only if, the restricted morphism $f|_{\mathbf{k}_{S_\alpha}}^{\mathbf{k}_{T_\beta}}\colon \mathbf{k}_{S_\alpha} \to \mathbf{k}_{T_\beta}$ is nonzero: see the left-hand side of Figure 2.3 for an example. This representation can be seen as a continuous version of the commutative diagrams (2.2) for morphisms of persistence modules. ■

2.2.3 Matrix reduction and the pairing uniqueness lemma

In this section, we consider the persistent homology of filtered simplicial complexes. In this context, matrix representations appear naturally in the persistent homology pipeline when passing from filtered (simplicial) chain complexes (Definition 2.8) to persistent homology. Observe that a filtered chain complex (C_*, ∂) can be viewed equivalently as a chain complex of persistence modules. Persistent homology then fits in the short exact sequence of persistence modules (Definition 2.52)

$$0 \to C_{d+1} \xrightarrow{\partial_{d+1}} \ker \partial_d \twoheadrightarrow H_d \to 0.$$

Note that C_{d+1} and $\ker \partial_d$ are free modules: once a simplex is added as a d-chain, it never vanishes.

Now consider a matrix representation A of ∂_{d+1} and write $C_d = \bigoplus_{i=1}^m \mathbf{k}_{[s_i,\infty)}$ and $C_{d+1} = \bigoplus_{j=1}^n \mathbf{k}_{[t_j,\infty)}$. Suppose that a subset $I \subseteq \{1, \ldots, m\}$ of rows generate $\ker \partial_d \subseteq C_d$ and that the matrix A has at most one nonzero coefficient in each row and column. In this case, the restricted morphism $\partial_{d+1}|^{\ker \partial_d}$ decomposes as

$$\bigoplus_{\substack{i \in I \\ j, A_{i,j} \neq 0}} (\mathbf{k}_{[t_j,\infty)} \to \mathbf{k}_{[s_i,\infty)}) \oplus \bigoplus_{\substack{i \in I \\ A_{i,*} = 0}} (0 \to \mathbf{k}_{[s_i,\infty)}),$$

where each summand is seen as a morphism from C_{d+1} to $\ker \partial_d$. The cokernel of this map is easy to compute, and we deduce that the persistent homology is the direct sum of bars

$$H_d = \bigoplus_{\substack{i \in I \\ j, A_{i,j} \neq 0}} \mathbf{k}_{[s_i,t_j)} \oplus \bigoplus_{\substack{i \in I \\ A_{i,*} = 0}} \mathbf{k}_{[s_i,\infty)},$$

which directly gives the barcode.

As it turns out, there is an algorithm for reducing matrix representations to this ideal form. Consider Algorithm 2.1 where, given a matrix $A \in \mathbf{k}^{m \times n}$, $A_{i,*}$ denotes the i^{th} row of A, $A_{*,j}$ denotes the j^{th} column of A, and $\text{low}(j) := \max\{i \geq 1 \mid A_{i,j} \neq 0\}$ denotes the index of the lowest nonzero element of $A_{*,j}$.

In the first while loop, we reduce A so that its lowest nonzero coefficient is not on the same row as the lowest nonzero coefficient of a previous column. In the second while loop, we clear all nonzero coefficients above the lowest one. We check that both types of operations — left-to-right column addition and

Algorithm 2.1: Standard reduction algorithm.

Input: matrix $A \in \mathbf{k}^{m \times n}$ representing a morphism $f\colon V \to W$ of free tame modules.

for $j = 1, \ldots, n$ **do**
  while $\exists j' < j$, $\mathrm{low}(j') = \mathrm{low}(j)$ **do**
    $\lambda \leftarrow A_{\mathrm{low}(j),j} / A_{\mathrm{low}(j'),j'}$;
    $A_{*,j} \leftarrow A_{*,j} - \lambda A_{*,j'}$;
  end
  while $\exists i' < i$, $A_{i',j} \neq 0$ **do**
    $\lambda \leftarrow A_{i',j} / A_{i,j}$;
    $A_{i',*} \leftarrow A_{i',*} - \lambda A_{i,*}$;
  end
end
return A

bottom-to-top row addition — return valid matrix representations. Indeed, adding column j' to j with $j' < j$ corresponds to multiplying the matrix A on the right by the matrix $(I_n + E_{j',j})$, where $E_{j',j}$ is the matrix with all 0's except for a 1 in position (j', j). Since the rows and columns of this matrix are ordered increasingly by their label's birth time, and $j' < j$, we deduce that we can have a nonzero coefficient at (j', j). We reason similarly for bottom-to-top row operations.

The output matrix has at most one nonzero coefficient in each row and column, and while the algorithm does not specify exactly in which order to do matrix reductions, we have the following fundamental result:

2.51 **Theorem** (Pairing uniqueness lemma, [18, 36]). *The output of Algorithm 2.1 does not depend on the ordering choices of the while loops. In other words, it does not matter how the columns j' and rows i' are chosen.*

Moreover, applying Algorithm 2.1 to a matrix representation of ∂_d will also give a basis of $\ker \partial_d$, namely the generators whose columns are 0. In this way, applying Algorithm 2.1 to the boundary maps of a filtered chain complex gives the barcode of the persistent homology as follows: given the reduced matrix of ∂_{d+1} and a row i,

- if there is a column j such that $\mathrm{low}(j) = i$, then we have a bar starting at the grade of row i and ending at the grade of column j;

- otherwise, if the i^{th} column of the reduced matrix of ∂_d is zero, then we have a bar starting at the grade of row i and ending at $+\infty$.
- otherwise, row i ends a bar in ∂_d but does not create a bar.

Matrix reduction is a key computational tool that we use in Papers B and C. In the latter paper, we also generalize the reduction algorithm to p.f.d. modules.

2.3 Persistent homological algebra

In this section, we recall the homological algebraic theory for persistence modules, considered as functors over posets.

2.3.1 Projective modules

Let $(I, \leq)$ be a finite poset. Recall from Definition 2.44 and Remark 2.49 that a free module is a direct sum of persistence modules of the form $\mathbf{k}_{\{I \geq s\}}$ for some s in I. The term *free* is justified by considering $\mathbf{k}_{\{I \geq s\}}$ to be the composition of the following functors

$$\begin{cases} I & \to & \mathbf{Set} & \to & \mathbf{Vect} \\ t & \mapsto & \mathrm{mor}_I(s,t) & \mapsto & \mathbf{k}\,\mathrm{mor}_I(s,t), \end{cases}$$

where the second functor is the functor that takes a set and returns a vector space of formal linear combinations of the set elements (Example 2.13).

2.52 **Definition.** A sequence of morphisms $U \xrightarrow{f} V \xrightarrow{g} W$ is **exact** if the following sequence of vector spaces

$$\hom(F,U) \xrightarrow{\hom(F,f)} \hom(F,V) \xrightarrow{\hom(F,g)} \hom(F,W)$$

is exact for every free functor F. That is, the image of the linear map $\hom(F,f)$ that post-composes with f is the kernel of the map $\hom(F,g)$ that post-composes with g. A longer sequence of morphisms $\cdots \to V_{d+1} \to V_d \to V_{d-1} \to \cdots$ is **exact** if every pair of consecutive morphisms is exact.

Note that a morphism $f\colon V \to W$ is an monomorphism (resp. epimorphism) if, and only if, the sequence

$$0 \to V \xrightarrow{f} W \qquad (\text{resp. } V \xrightarrow{f} W \to 0)$$

is exact.

A persistence module F is **projective** if, for every exact sequence $U \xrightarrow{f} V \xrightarrow{g} W$, the following sequence of vector spaces is exact:

$$\hom(F,U) \xrightarrow{\hom(F,f)} \hom(F,V) \xrightarrow{\hom(F,g)} \hom(F,W)$$

It turns out that all projectives are free [Paper A, Proposition 3.4].

Let V be a persistence module. A **projective cover of** V is an epimorphism $F \twoheadrightarrow V$ where F is projective. This cover is **minimal** if all of its endomorphisms (morphisms $F \to F$ that commute with the epimorphism $F \twoheadrightarrow V$) are isomorphisms. The minimal cover exists and is unique up to isomorphism [Paper A, §2.4]. A **projective resolution of** V is an exact sequence

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} V \to 0$$

where the F_d, $d \geq 0$, are projective. Similarly to covers, a resolution is **minimal** if its endomorphisms (morphisms $F_* \to F_*$ as chain complexes) are isomorphisms, and this minimal resolution exists and is unique up to isomorphism. ■

2.53 **Remark.** One way of presenting a persistence module V is by the first two terms of a projective resolution. Indeed, given a projective resolution $(F_d)_{d\geq 0}$ of V, the cokernel of the map $\partial_0 \colon F_1 \to F_0$ is isomorphic to V. Since F_0 and F_1 are free, we can construct generating sets (Definition 2.44). For example, writing $F_0 = \bigoplus_{\alpha\in A} \mathbf{k}_{[s_\alpha,\infty)}$, we have the immediate generating set $\{\mathbf{k}_{[s_\alpha,\infty)} \hookrightarrow F_0\}_{\alpha\in A}$. We call this generating set of F_0 the set of **generators of** V, and the analogous generating set of F_1 the set of **relations of** V. The matrix representation of ∂_0 is called a **presentation matrix of** V. ■

2.3.2 Betti diagrams and local Koszul complexes

Minimal projective resolutions provide an alternative to the barcode decomposition theorem that does not hold for arbitrary posets. Indeed, as alluded to in Section 2.2.3, barcode decompositions can be expressed as resolutions

$$0 \to \bigoplus_j \mathbf{k}_{[t_j,\infty)} \to \bigoplus_i \mathbf{k}_{[s_i,\infty)} \to V \to 0.$$

One way of extracting numeric invariants from more general minimal resolutions is to consider the multiplicities of free modules in the resolution.

2.54 **Definition.** Let V be a persistence module over $(I, \leq)$, (F_*, ∂) its minimal projective resolution, and $d \geq 0$ an integer. Recall that all projectives are free: thus the minimal resolution of V looks like

$$\cdots \to \bigoplus_{t \in I} \mathbf{k}_{\{I \geq t\}}^{\oplus \beta^1(t)} \to \bigoplus_{s \in I} \mathbf{k}_{\{I \geq s\}}^{\oplus \beta^0(s)} \to V \to 0.$$

The $d^{\mathbf{th}}$ **Betti diagram** is the function $\beta^d \colon I \to \mathbf{N}$ that gives the multiplicity of each $\mathbf{k}_{\{I \geq s\}}$, $s \in I$, in the free module F_d. ■

The naive way of computing Betti diagrams is the computation of the entire minimal resolution of V and then extracting the multiplicities. There is a more efficient way in the case where the indexing poset I is minimally well-behaved. We present this strategy over the field of two elements $\mathbf{F}_2$ for simplicity; the strategy works over every field with extra signs in the constructions [Paper A, §3.7].

2.55 **Definition.** Let $(I, \leq)$ be a finite poset and t an element of I. A **parent** of I is an element $s < t$ such that there is no element u satisfying $s < u < t$. We denote the set of parents of t by $\mathscr{U}(t)$.

Now suppose that I is an **upper semilattice**, that is, every pair of elements s, t admits a minimal upper bound $s \wedge t$ called the **meet**. Let V be a persistence module over I and t an element of I. The **(local) Koszul complex of** V **at** t is a chain complex of vector spaces $(\mathscr{K}_t V)$ defined as follows. For all $d \geq 0$, we define

$$(\mathscr{K}_t V)_d \coloneqq \begin{cases} V(t) & \text{if } d = 0, \\ \displaystyle\bigoplus_{\substack{S \subseteq \mathscr{U}(t), |S| = d \\ S \text{ has lower bound}}} V(\textstyle\bigwedge S) & \text{otherwise.} \end{cases}$$

For example, $(\mathscr{K}_t V)_1 = \bigoplus_{s \in \mathscr{U}(t)} V(s)$. The boundary maps are defined as follows:

- Define $\partial_1 \colon \bigoplus_{s \in \mathscr{U}(t)} V(s) \to V(t)$ as the linear map $\sum_{s \in \mathscr{U}(t)} V(s \leq t)$.
- For $d > 0$, define $\partial_d \colon (\mathscr{K}_t V)_d \to (\mathscr{K}_t V)_{d-1}$ as the sum, for each subset $S = \{s_0, \cdots, s_d\} \subseteq \mathscr{U}(t)$ of size d and s in S, of the transition maps $V(\bigwedge S) \to V(\bigwedge S \setminus \{s\})$. ■

The Betti diagrams of a persistence module V can then be computed as the dimensions of the homology of the Koszul complex:

2.56 **Theorem** (Paper A, Theorem 3.8). *Let I be a finite upper semilattice, t an element of I. Then, for every persistence module V over I and every $d \geq 0$, the following equality holds:*

$$\beta^d V(t) = \dim H_d(\mathscr{K}_t V)$$

This result is extended to the case of relative homological algebra in Paper A: see Chapter 4 for details.

2.4 Metric persistent algebra

In this section we continue the study of distances on persistence modules introduced in Section 2.1.9.

2.4.1 Algebraic Wasserstein distances

Here we present a generalization of the isometry between interleaving and bottleneck distances (Theorem 2.37) to p-Wasserstein distances (Definition 2.31). The p-Wasserstein distance, like the bottleneck distance, is defined combinatorially on the space of persistence diagrams. We are interested in characterizing this distance algebraically.

There is a strategy for algebraically defining pseudometrics on persistence modules via so-called **noise systems**. In particular, this method works for general persistence modules, where we do not have barcode decompositions. The strategy decides which persistence modules will be considered as "small".

2.57 **Definition** ([28, 45]). Let $(I, \leq)$ be a poset. A **noise system** is a sequence $\mathscr{S} = \{\mathscr{S}_\varepsilon\}_{\varepsilon \in [0,\infty)}$ of collections of persistence modules such that

1. $0 \in \mathscr{S}_\varepsilon$ for all $\varepsilon \in [0, \infty)$,
2. $\mathscr{S}_\delta \subseteq \mathscr{S}_\varepsilon$ for all $\delta \leq \varepsilon$, and
3. for all short exact sequences $0 \to V_0 \to V_1 \to V_2 \to 0$,
    - if V_1 is in $\mathscr{S}_\varepsilon$, then V_0 and V_2 are in $\mathscr{S}_\varepsilon$;
    - if V_0 is in $\mathscr{S}_\delta$ and V_2 is in $\mathscr{S}_\varepsilon$, then V_1 is in $\mathscr{S}_{\delta+\varepsilon}$.

For V a persistence module, we call $\alpha(V) := \inf\{\varepsilon \geq 0 \mid V \in \mathscr{S}_\varepsilon\}$ the **amplitude of** V.

Let $\varepsilon \geq 0$ be a number. Two persistence modules V and W are **ε-close** if there exist a persistence module U and a pair of morphism $V \xleftarrow{f} U \xrightarrow{g} W$ (called a **span**) such that

$$\alpha(\ker f) + \alpha(\operatorname{coker} f) + \alpha(\ker g) + \alpha(\operatorname{coker} g) \leq \varepsilon. \tag{2.5}$$

We then define the pseudometric induced by $\mathscr{S}$ to be

$$d_{\mathscr{S}}(V,W) := \inf\{\varepsilon \geq 0 \mid V \text{ and } W \text{ are } \varepsilon\text{-close}\}. \qquad \blacksquare$$

2.58 **Definition.** Let $p \in [1,\infty]$ be a number. Given a p.f.d. persistence module $V = \bigoplus_{\alpha\in A} \mathbf{k}_{S_\alpha}$, we define the ***p*-norm of** V to be the nonnegative number

$$\|V\|_p := \left\| \left(t(S_\alpha) - s(S_\alpha) \right)_{\alpha\in A} \right\|_p,$$

that is, the p-norm of the vector of lengths of the intervals S_α.

We then define the noise system (Definition 2.57) $\mathscr{S}^p$ where, for all $\varepsilon \geq 0$,

$$\mathscr{S}^p_\varepsilon := \{V \in \mathbf{Pfd} \mid \|V\|_p \leq \varepsilon\}.$$

In other words, the amplitude of a persistence module is its p-norm. Following Definition 2.57, this defines a distance

$$d_{\mathscr{S}^p}(U,V) := \inf\left\{ \varepsilon \geq 0 \;\middle|\; \begin{array}{l} \exists U \xleftarrow{f} W \xrightarrow{g} V, \\ \|\ker f\|_p + \|\operatorname{coker} f\|_p + \|\ker g\|_p + \|\operatorname{coker} g\|_p \leq \varepsilon \end{array} \right\}. \qquad \blacksquare$$

To check that this is indeed a distance, we need to check the noise system axioms.

2.59 **Theorem** (Paper B, Theorem 4.16). *$\mathscr{S}^p$ is indeed a noise system, for all $p \in [1,\infty]$. As a consequence, $d_{\mathscr{S}^p}$ is a pseudometric on p.f.d. persistence modules.*

A similar statement of this result appears in [46] (version 5 specifically of this arXiv submission) as Theorems 7.27 and 7.28.

2.4.2 Induced matchings

As we have seen in Definition 2.31, matchings of barcodes and persistence diagrams play an important role in the construction of bottleneck and Wasserstein distances. Computing matchings between barcodes of datasets has also been explored in the training of topologically robust image segmentation models [48, 49]. One natural way of constructing matchings between barcodes is to induce them from morphisms between persistence modules. In this section, we will present two methods of inducing matchings from morphisms: the $\mathscr{X}$-**matching** by Bauer and Lesnick [3] and the $\mathscr{P}$-**matching** by Gonzalez-Diaz, Soriano-Trigueros, and Torras-Casas [29].

2.60 **Notation** ([3, §2.3]). We denote intervals of $\mathbf{R} \cup \pm\infty$ as follows: for all s, t in $\mathbf{R} \cup \pm\infty$, we write

$$\langle s^-, t^- \rangle := [s, t), \qquad \langle s^-, t^+ \rangle := [s, t],$$
$$\langle s^+, t^- \rangle := (s, t), \qquad \langle s^+, t^+ \rangle := (s, t].$$

When the inclusion of the endpoint is unknown, we write $s^{\pm}$. ■

2.61 **Definition** ([3]). Let $f\colon V \hookrightarrow W$ be a monomorphism of p.f.d. modules over $\mathbf{R}$. Recall that the barcodes $\mathscr{B}(V)$ and $\mathscr{B}(W)$ consist of intervals of $\mathbf{R}$ that may be open or closed on either end. In this definition, we consider these barcodes to be sets instead of multisets: intervals that appear multiple times are formally distinguished (for example, by adding an indexing to each interval).

For all t in $\mathbf{R}$ and $\tau \in \{-, +\}$, consider the submultiset $(\langle r_1^{\pm}, t^{\tau} \rangle, \ldots, \langle r_m^{\pm}, t^{\tau} \rangle)$ of bars in $\mathscr{B}(V)$ whose end-point is t excluded (if $\tau = -$) or included (if $\tau = +$), sorted by increasing start-point (with s^- smaller than s^+). Similarly, consider $(\langle s_1^{\pm}, t^{\tau} \rangle, \ldots, \langle s_n^{\pm}, t^{\tau} \rangle)$ in $\mathscr{B}(W)$. It turns out that $m \leq n$ [3, Theorem 4.2], and so we can construct a matching

$$\mathscr{X}_{t^{\tau}} := \{(\langle r_i^{\pm}, t^{\tau} \rangle, \langle s_i^{\pm}, t^{\tau} \rangle) \mid 1 \leq i \leq m\}.$$

The $\mathscr{X}$-**matching of the monomorphism** f, denoted by $\mathscr{X}(f)$, is defined as the union

$$\mathscr{X}(f) := \bigcup_{t \in \mathbf{R}, \tau \in \{-,+\}} \mathscr{X}_{t^{\tau}} \in \mathbf{Match}(\mathscr{B}(V), \mathscr{B}(W)).$$

We dually define the $\mathscr{X}$-matching $\mathscr{X}(f)$ for an epimorphism $f\colon V \twoheadrightarrow W$, where we sort bars in $\mathscr{B}(V)$ and $\mathscr{B}(W)$ by start-point instead of end-point.

Now let $f\colon V \to W$ be an arbitrary morphism of p.f.d. modules, and consider $f = m \circ e$ an **epi-mono factorization**: that is, e is an epimorphism, m is a monomorphism, and we have the composition

$$f\colon V \xrightarrow{e} \operatorname{im}(f) \xrightarrow{m} W.$$

Note that this epi-mono factorization is only unique up to isomorphism on $\operatorname{im}(f)$. Then the **$\mathscr{X}$-matching of** f is

$$\begin{aligned}\mathscr{X}(f) &:= \mathscr{X}(m) \circ \mathscr{X}(e)\\ &:= \{(a,c) \mid \exists b \in \mathscr{B}(\operatorname{im}(f)),\, (a,b) \in \mathscr{X}(e), (b,c) \in \mathscr{X}(m)\},\end{aligned}$$

and this matching does not depend on the choice of epi-mono factorization. ∎

The $\mathscr{X}$-matching is functorial on monomorphisms and epimorphisms [3, Proposition 5.7] but is not additive. That is, $\mathscr{X}(f \oplus g)$ is not necessarily equal to $\mathscr{X}(f) \sqcup \mathscr{X}(g)$. Moreover, it does not really conserve information about the underlying morphism, especially for mono- and epimorphisms: it only requires the existence of such a morphism in order to be defined; the matchings themselves just match longest bars to longest bars. These limitations were a motivation for the definition of $\mathscr{P}$-matchings of [29]:

2.62 **Definition** ([29]). Let $f\colon V \to W$ be a morphism of tame modules with fixed bases and let A be the ordered matrix representation of f where the columns are ordered lexicographically[6] by start- then end-point and the rows by end- then start-point. We apply Algorithm 2.2 to A: the output matrix has unique pivots, and we define the **$\mathscr{P}$-matching of** f to be the matching $\mathscr{P}(f)$ in $\mathbf{Match}(\mathscr{B}(V), \mathscr{B}(W))$ consisting of the pairs of labels of the pivots' rows and columns. ∎

As opposed to the $\mathscr{X}$-matching, the $\mathscr{P}$-matching is additive but not functorial on mono- or epimorphisms. In fact, we show in Proposition C.40 that an operator that induces matchings from morphisms cannot be both functorial on mono- and epimorphisms and additive.

[6] The **lexicographic order** $\leq_{\text{lex}}$ is given by $(a,b) \leq_{\text{lex}} (a',b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$.

Algorithm 2.2: Reduction algorithm from [29].

Input: matrix $A \in \mathbf{k}^{m\times n}$ representing a morphism $f\colon V \to W$ of tame modules.

for $j = 1, \dots, n$ **do**
    while $\exists j' < j$, $\mathrm{low}(j') = \mathrm{low}(j)$ **do**
        $\lambda \leftarrow A_{\mathrm{low}(j),j}/A_{\mathrm{low}(j'),j'}$;
        $A_{*,j} \leftarrow A_{*,j} - \lambda A_{*,j'}$;
        for $i = 1, \dots, m$ **do**
            if *label of column j does not overlap label of row i above* **then**
                $A_{i,j} \leftarrow 0$;
            end
        end
    end
end
return A

In Papers B and C, we study other methods of inducing matchings from morphisms. In particular, we introduce the notion of **bar-to-bar morphisms** in Paper C, which we use to construct induced matchings. See Chapter 4 for more details.

3 Morse theory and metric algebraic geometry

When the blackbird flew out of sight,
It marked the edge
Of one of many circles.

In the previous chapter, we presented a general theory for algebraically studying filtered spaces using the tools of persistent homology. In this chapter, we present tools from a slightly older theory, namely Morse theory, for studying filtered manifolds. We cover preliminaries on Morse theory and related studies of critical points of distance functions in metric algebraic geometry.

3.1 A combinatorial encoding of topology

3.1 **Definition.** Let X be a smooth manifold and $f\colon X \to \mathbf{R}$ a smooth function. A **critical point of** f is a point x of X where the differential $\partial_x f$ of f is 0. A **critical value** is the image $f(x)$ of a critical point.

The **Hessian of** f **at** x is the matrix of second partial derivatives of f at x, denoted by $H(f)(x)$. A critical point x of X is **nondegenerate** if $H(f)(x)$ is nonsingular, and **degenerate** otherwise. Recall that the Hessian is a symmetric real matrix, and therefore is diagonalizable. The **index** of a nondegenerate critical point is the number of negative eigenvalues of its Hessian. A **Morse function** is a smooth function on a smooth manifold that has no degenerate critical points. ■

The index of a nondegenerate critical point has the following interpretation in terms of normal forms.

3.2 **Proposition** ([39, Lemma 2.2]). *Let X be a smooth manifold of dimension n, $f\colon X \to \mathbf{R}$ a smooth function, and $x \in X$ a nondegenerate critical point with index $\iota \in \{0,\dots,n\}$. Then there exists a neighborhood U of x and a homeomorphism $\psi\colon \mathbf{R}^n \to U$ such that*

$$f(\psi(t_1,\dots,t_n)) = f(x) - t_1^2 - \cdots - t_\iota^2 + t_{\iota+1}^2 + \cdots + t_n^2. \tag{3.1}$$

Viewing the filtration $f\colon X \to \mathbf{R}$ as a height function and considering the **normal form** (3.1), we can then interpret the nondegenerate critical point x as a saddle point of X where the manifold slopes downwards in ι directions and upwards in $n-\iota$ directions.

This algebraic encoding of the geometry of a filtered manifold can be developed further by the **Morse lemmas**, which we present as a theorem:

3.3 **Theorem** ([39, Theorems 3.1 & 3.2]). *Let X be a smooth manifold, $f\colon X \to \mathbf{R}$ a smooth function, and $s \le t$ in $\mathbf{R}$. Recall the notation $X_{\le t} := f^{-1}((-\infty,t])$.*

1. *If the interval $[s,t]$ contains no critical values, then $X_{\le s}$ is diffeomorphic to $X_{\le t}$ and $X_{\le t}$ deformation retracts onto $X_{\le s}$. In terms of homology, we get $H_*(X_{\le s}) \cong H_*(X_{\le t})$.*
2. *Suppose that there is a unique nondegenerate critical point $x \in X$ whose value $f(x)$ is in the interval $[s,t]$, and let ι be the index of x. Then $X_{\le t}$ is homotopy equivalent to $X_{\le s}$ with an ι-cell attached (i.e., an ι-disk glued to $X_{\le s}$ along their boundaries). In terms of homology, this means that*

$$H_k(X_{\le t}) \cong \begin{cases} H_k(X_{\le s}) \oplus \mathbf{k} & \text{if } k = \iota, \\ H_k(X_{\le s}) & \text{otherwise.} \end{cases}$$

3.4 **Example.** Consider Figure 3.1, where we equip a torus with a height map as its filtration. In this case, there are four critical points. From bottom to top

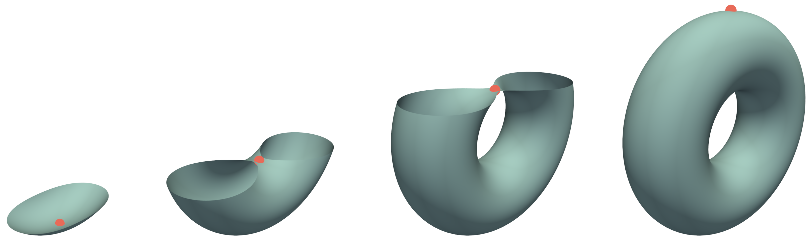


Figure 3.1: Sublevel sets of a torus filtered by a height map, after passing each of its four critical points (marked in orange).

(following the filtration), these critical points have index 0, 1, 1, and 2, respectively, and we can see that passing from one sublevel set corresponds to the attachment of a cell of corresponding dimension. ■

In addition, we have the following homological results.

3.5 **Definition.** Let X be a smooth manifold. For all integers $d \geq 0$, we call $b_d(X) \coloneqq \dim H_d(X)$ the d^{th} **Betti number of** X. ■

3.6 **Theorem** ([39, §5]). *Let X be a smooth manifold and $f\colon X \to \mathbf{R}$ a smooth function whose critical points are isolated and nondegenerate. For all integers $d \geq 0$, denote by C^d the number of critical points of f of index d. Then, for all $d \geq 0$, we have the inequality*

$$C^d - C^{d-1} + \cdots + (-1)^d C^0 \geq b_d(X) - b_{d-1}(X) + \cdots + (-1)^d b_0(X).$$

In particular, for all $d \geq 0$, we get

$$C^d \geq b_d(X).$$

When the set of critical points is easy to compute, this gives powerful upper bounds on the homology of the manifold X.

3.2 Morse theory for continuous selections

Morse theory was originally developed for smooth function on smooth manifolds. However, it has been generalized in many directions, including Morse-Bott theory [2], stratified Morse theory [30], and discrete Morse theory [27].

In Paper D we develop Morse theory for continuous selections, as presented in [1]. We recall the original theory here.

3.7 **Definition.** Let X be a smooth manifold and $\{f_0,\dots,f_m\} \subseteq \mathscr{C}^2(X,\mathbf{R})$ a set of $\mathscr{C}^2$ functions. A **continuous selection from** $\{f_0,\dots,f_m\}$ is a continuous function $f\colon X \to \mathbf{R}$ such that, for every x in X, there exists i in $\{0,\dots,m\}$ such that $f(x) = f_i(x)$. We denote by $\mathrm{CS}(f_0,\dots,f_m)$ the collection of all continuous selections from $\{f_0,\dots,f_m\}$.

Let $f \in \mathrm{CS}(f_0,\dots,f_m)$ be a continuous selection. The **subdifferential** of f at a point $x \in X$ is (see [1] or Paper D for a more general definition)

$$\partial f(x) := \mathrm{conv}\{\nabla f_i(x) \mid i \in I(x)\},$$

where $I(x)$ is the set of indices $i \in \{0,\dots,m\}$ such that $x \in \overline{\mathrm{int}\{f(y) = f_i(y)\}}$, i.e., x is in the closure of the interior of the set where f is equal to f_i.

A point x in X is a **critical point of** f if 0 belongs to its subdifferential $\partial f(x)$. A critical point $x \in X$ is **nondegenerate** if the following two conditions hold:

1. For every $i \in I(x)$, the set of differentials $\{D_x f_j \mid j \in I(x) \setminus \{i\}\}$ is linearly independent.
2. Denote by $(\lambda_i)_{i\in I(x)}$ the unique list of nonnegative real numbers such that

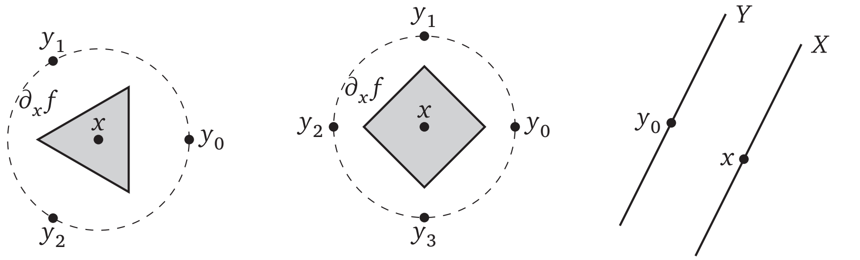
$$\sum_{i\in I(x)} \lambda_i = 1 \quad \text{and} \quad \sum_{i\in I(x)} \lambda_i D_x f_i = 0$$

and by $\lambda f\colon X \to \mathbf{R}$ the function $\lambda f(y) := \sum_{i\in I(x)} \lambda_i f_i(y)$. Then the second differential of λf_i is nondegenerate on

$$V(x) := \bigcap_{i\in I(x)} \ker(D_x f_i).$$

We write $k(x) := \#I(x) - 1$ and call this number the **piecewise linear index of** x. We denote by $\iota(x)$ the number of negative eigenvalues of the restriction of the second differential of λf to $V(x)$, and call it the **quadratic index of** x. See Figure 3.2 for an illustration of nondegenerate and degenerate critical points of $\mathrm{dist}_Y|_X$ and Figure 3.3 for an illustration of different indices for $\mathrm{dist}_Y|_X$. ■

For a nondegenerate critical point, we recover the second Morse lemma: writing $k = k(x)$ and $\iota = \iota(x)$, there exists a neighborhood U of x and a locally Lipschitz

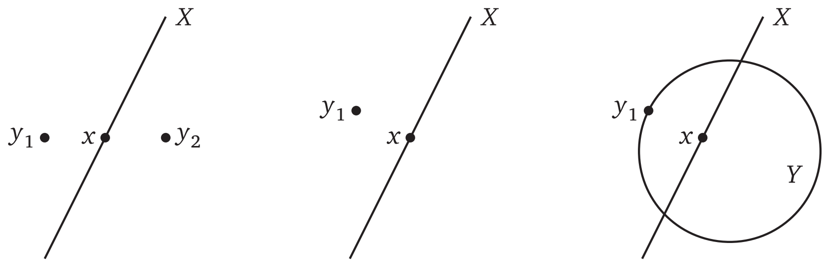


(a) nondegenerate critical point

(b) degenerate critical point for the first condition

(c) degenerate critical point for the second condition

Figure 3.2: Examples of critical points of $f = \mathrm{dist}$ and their degeneracy. (a) and (b): X is the plane $\mathbf{R}^2$ and Y is a finite set of points in the plane. (c): X and Y are lines.

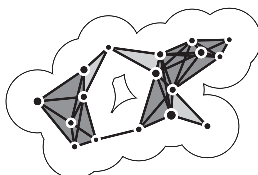


(a) indices $(1,0)$

(b) indices $(0,0)$

(c) indices $(0,1)$

Figure 3.3: Examples of dist and $x \in X$ with different piecewise linear and quadratic indices, written as $(k(x), \iota(x))$. (a) and (b): X is a line and Y is a set of points. (c): X is a line, Y is a circle, and y_1 is the closest point of Y to x.



homeomorphism $\psi\colon \mathbf{R}^k \times \mathbf{R}^{n-k} \to U$ (where n is the dimension of the manifold X) such that

$$f(\psi(t_1,\dots,t_n)) = f(x) + \ell(t_1,\dots,t_k) - \sum_{j=k+1}^{k+\iota} t_j^2 + \sum_{j=k+\iota+1}^{n} t_j^2,$$

where $\ell(t) \in CS(t_1,\dots,t_k,-(t_1+\dots+t_k))$ is a continuous selection of linear functions, making it piecewise linear. From this result, the rest of Morse theory follows: see [1] for the general theory and Paper D for results specific to distance functions.

3.3 Distance functions between algebraic varieties

For algebraic spaces, the notion of critical points of distance functions has been studied extensively in the past few decades, as part of the growing field of **metric algebraic geometry** [11, 51].

3.8 **Definition.** A **real algebraic variety** is the set of solutions of a system of multivariate polynomial equations over the real numbers.

Consider the space $\mathbf{R}^n$ for some $n \geq 1$ and an integer $d \geq 0$. Denote by $\mathscr{P}_{n,d}$ the space of real polynomials in n variables, of degree d. Given polynomials $\vec{p} = p_1,\dots,p_k$ in $\mathscr{P}_{n,d}$, we denote by $V(\vec{p})$ the algebraic variety that is the set of solutions of the equations $p_i(x) = 0$, for all $i \in \{1,\dots,k\}$.

The **dimension** of a variety X is its dimension around any nonsingular point, seen as a manifold. We denote it by $\dim X$. Its **codimension** is $\operatorname{codim} V := n - \dim X$ where n is the dimension of the ambient real or complex space.

An algebraic variety X is a **complete intersection** if there exist $\operatorname{codim} V$ many polynomials $p_1,\dots,p_{\operatorname{codim} X}$ such that $X = V(p_1,\dots,p_{\operatorname{codim} X})$. ■

Algebraic geometers are often interested in integer invariants for algebraic varieties, and metric algebraic geometers are no exception. We present two of these **degrees**:

3.9 **Definition.** Let X be a real algebraic variety and u an arbitrary point in $\mathbf{R}^n$. Consider the function

$$\operatorname{dist}_u\colon \begin{cases} X & \to & [0,\infty) \\ x & \mapsto & \|u-x\|_2^2 \end{cases}$$

defined as the squared Euclidean distance. It turns out that the number of critical points of dist_u is constant for a dense open subset of choices of $\mathbf{R}^n$: the **Euclidean distance degree (EDD) of** X is then this constant number [23]. An interpretation of the EDD is as a measure of the complexity of finding the closest point on X to a given point u in $\mathbf{R}^n$.

The **bottlenecks of** X are pairs (x, y) of distinct points of X such that the line passing through x and y is normal (orthogonal) to X at both points. Equivalently, bottlenecks are the nontrivial critical points of the squared distance function on X, that is, the critical points of the function

$$\text{dist}\,|_X : \begin{cases} X \times X & \rightarrow & [0, \infty) \\ (x, y) & \mapsto & \|x - y\|_2^2, \end{cases}$$

with the extra condition that $x \neq y$. The **bottleneck degree (BND) of** X is the number of bottlenecks, under some regularity conditions on X [21]. ∎

Both the Euclidean distance degree and the bottleneck degree fit within the framework of critical points of the distance function from an algebraic variety, restricted to another algebraic variety. In Papers D and E, we study this generalized notion for generic varieties.

4 Papers and contributions

The blackbird whirled in the autumn winds.
It was a small part of the pantomime.

A Koszul complexes and relative homological algebra of functors over posets

In this paper, we study persistence modules over a finite poset $(I, \leq)$ using the functorial perspective (Section 2.1.5). We extend the results of Section 2.3 to the **relative homological case**. Instead of approximating functors over I by projective resolutions, where each term is a sum of free functors, we approximate them by other "simple" functors. Given a collection $\mathscr{P}$ of such simple functors, we define $\mathscr{P}$-exactness, $\mathscr{P}$-projectives, $\mathscr{P}$-covers, and $\mathscr{P}$-resolutions, which are the relative versions of the "absolute" notions defined in Section 2.3. A **$\mathscr{P}$-free functor** is a functor that is isomorphic to a direct sum of elements of $\mathscr{P}$. The collection $\mathscr{P}$ is then **acyclic** if every $\mathscr{P}$-projective is $\mathscr{P}$-free. Under this condition, we can then define **Betti diagrams relative to** $\mathscr{P}$: given a functor $M\colon I \to \mathbf{Vect}$ and an element $a \in J$ such that $\mathscr{T}(a) \neq 0$, we denote by $\beta^d_{\mathscr{P}} M(a)$ the multiplicity of $\mathscr{T}(a)$ in the d^{th} term of the minimal $\mathscr{P}$-projective resolution of M.

The principal result of this paper is that, under reasonable conditions on $\mathscr{P}$, we can compute the relative Betti diagrams using local Koszul complexes. First, assume that the collection $\mathscr{P}$ is parameterized by a functor $\mathscr{T}\colon (J,\preceq)^{\mathrm{op}} \to \mathrm{Fun}(I,\mathbf{Vect})$ where $(J,\preceq)$ is a finite upper semilattice (see Paper A, §3.6) and $\mathscr{P} = \{\mathscr{T}(a) \mid \mathscr{T}(a) \neq 0,\ a \in J\}$. The parameterization functor $\mathscr{T}$ is **thin** if, for all $a \preceq b$ in J, the space of natural transformation $\mathrm{Nat}(\mathscr{T}(b),\mathscr{T}(a))$ is of dimension at most 1. In this case, the collection $\mathscr{P}$ is acyclic, and so we can talk about Betti diagrams relative to $\mathscr{P}$. Given a functor $M\colon I \to \mathbf{Vect}$, we define the functor indexed by J

$$\mathscr{R}M\colon \begin{cases} J & \to & \mathbf{Vect} \\ a & \mapsto & \mathrm{Nat}(\mathscr{T}(a),M). \end{cases}$$

The main result is the following; compare with Theorem 2.56.

4.1 **Theorem** (Paper A, Corollary 5.20). *Let $(J,\preceq)$ be an upper semilattice and $\mathscr{T}$ a thin parameterization functor. Assume that*

1. *the set $\{a \in J \mid \mathscr{T}(a) = 0\}$ is closed under joins, and*
2. *for all a,b in J, if b is minimal $\succeq a$ such that $\mathrm{Nat}(\mathscr{T}(b),\mathscr{T}(a)) = 0$, then $\mathscr{T}(a) = 0$.*

Then, for all functors $M\colon I \to \mathbf{Vect}$, all a in J such that $\mathscr{T}(a) \neq 0$, and all $d \geq 0$,

$$\beta^d_{\mathscr{P}} M(a) = \dim H_d(\mathscr{K}_a \mathscr{R}M).$$

In other words, under the listed conditions, we can compute the Betti diagrams relative to $\mathscr{P}$ using the standard Koszul complexes defined on $\mathscr{R}M\colon J \to \mathbf{Vect}$.

Paper A ends with an extensive list of examples of collection $\mathscr{P}$ and parameterization functors $\mathscr{T}$ where the conditions of the theorem hold. This includes examples that have been studied in previous works, including **rectangles**, **lower hooks** [10], and **single-source spread modules** [8]. In fact, this work was in part motivated by the study of exact structures for persistence modules, in particular relative to lower hooks, in [10, Section 4]. We have also implemented the computation of Betti diagrams relative to lower hooks in Python [44]: see Figure 4.1 for a sample output.

Erratum

Thank you to Tada Shunsuke for pointing out the following mistake.

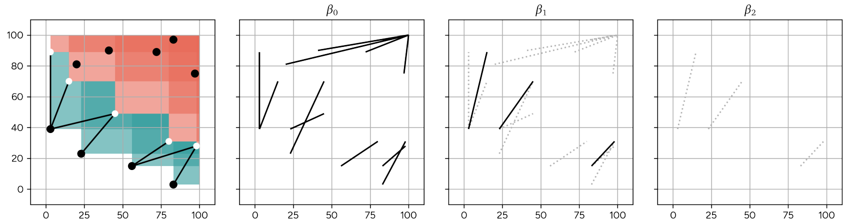


Figure 4.1: Sample output of `lowerhooks`. **Leftmost:** free presentation of a two-dimensional persistence module. Black points correspond to generators, white to relations, and lines correspond to nonzero coefficients in the presentation matrix (see Remark 2.53). **Right:** Betti diagrams relative to lower hooks. Lower hooks are parameterized by ordered pairs of points; a segment corresponds to a nonzero Betti number for the pair corresponding to the segment's endpoints. Dotted segments correspond to the Betti diagram one degree lower, for easier comparison between the degrees.

In §4.4, we write that a functor $F\colon J \to \mathbf{Vect}$ is isomorphic to a subfunctor of $\mathbf{k}_J$ if, and only if, F is a filtration and $\dim F(a) \leq 1$ for every a in J. This does not hold in all generality: consider the functor F defined by

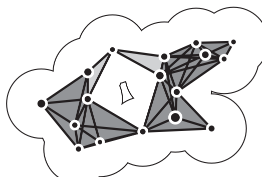
$$\begin{tikzcd} \mathbf{k} \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{k} \\ \mathbf{k} & \mathbf{k} \arrow[l, "1"] \arrow[u, "\lambda"'] \end{tikzcd}$$

where $\lambda \in \mathbf{k} \setminus \{0,1\}$. Then F is a filtration with pointwise dimension at most 1, but is not isomorphic to a subfunctor of $\mathbf{k}_J$. As a correction, we restrict the claim to the case where J is an upper semilattice. There is no impact on the rest of the paper.

Contributions

In this paper I co-wrote the main results, was the main contributor to the analysis of examples in the last section, and wrote the software `lowerhooks` [44] that applies our theoretical results to the specific case of lower hooks.

CRediT classification. Isaac Ren: conceptualization (equal); formal analysis (equal); software (lead); visualization (lead); writing – original draft preparation (equal); writing – review & editing (equal).



B Algebraic Wasserstein distances and stable homological invariants of data

In this paper, we define algebraic p-Wasserstein distances for tame persistence modules, proving that the sequence of collections $\mathscr{S} = \{\mathscr{S}_\varepsilon\}_{\varepsilon \geq 0}$ of Section 2.4.1 is indeed a noise system. This is an alternate proof to that of [46].

The difficult noise system axiom to prove is the exact sequence one (Definition 2.57), and specifically the following result (a simplified version of [Paper B, Lemma 4.15]):

4.2 **Proposition.** *Let* $0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$ *be an exact sequence of tame persistence modules, and* $p \in [1, \infty]$ *a number. Then*

$$\|V\|_p \leq \|U\|_p + \|W\|_p.$$

Our proof strategy is to observe that the morphism f is a monomorphism and that $W \cong \operatorname{coker} f$, and then study cases where the p-norm of $\operatorname{coker} f$ is easy to compute. We identify the following notion.

4.3 **Definition.** A morphism $f\colon V \to W$ of tame persistence modules is **bar-to-bar** when there exists a matrix representation of f such that each row and column has at most one nonzero coefficient. ■

In other words, for a bar-to-bar morphism, there exist barcode decompositions of the domain and codomain such that each bar of the domain maps to at most one bar of the codomain. The key result is the following.

4.4 **Theorem** (Paper B, Theorem 3.14). *Let* $f\colon V \hookrightarrow W$ *be a monomorphism of tame persistence modules. Then there exists a bar-to-bar monomorphism* $f_b\colon V \hookrightarrow W$ *such that* $\|\operatorname{coker} f_b\|_p \leq \|\operatorname{coker} f\|_p$ *for all* $p \in [1, \infty]$.

Proof sketch of Proposition 4.2. If $f\colon U \hookrightarrow V$ is bar-to-bar, then we know the barcode of $W \cong \operatorname{coker} f$ from the barcodes of U and V, and the inequality follows from the triangular inequality for usual p-norms.

For arbitrary $f\colon U \hookrightarrow V$, let $f_b\colon U \hookrightarrow W$ be a bar-to-bar morphism as in Theorem 4.4. Then

$$\|V\|_p \leq \|U\|_p + \|\operatorname{coker} f_b\|_p \leq \|U\|_p + \|W\|_p.$$

The first inequality comes from the previous observation, and the second from Theorem 4.4. □

In this paper, we also introduce additional parameterizations on p-Wasserstein distances. First, in the definition of ε-closeness, we can replace the sum in (2.5) with any q-norm, $q \in [1, \infty]$:

$$\|\alpha(\ker f), \alpha(\operatorname{coker} f), \alpha(\ker g), \alpha(\operatorname{coker} g)\|_q \leq \varepsilon.$$

Moreover, when taking the p-norm of a persistence module $V \cong \bigoplus_\alpha \mathbf{k}_{S_\alpha}$, we can apply a nondecreasing function $T\colon \mathbf{R} \to \mathbf{R}$ to intervals S_α before computing their length:

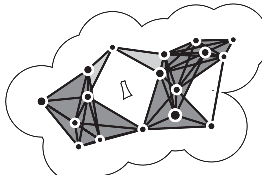
$$\|V\|_p \coloneqq \|(T(t(S_\alpha)) - T(s(S_\alpha)))\|_p.$$

This allows us to weight the importance of bars at different points of the indexing set $\mathbf{R}$. Given these parameterized pseudometrics, we are able to explicitly compute the **hierarchical stabilization** of persistence modules, which are real-valued functions that we can feed into machine learning frameworks [31, 45]. In the paper, we apply this analysis pipeline to classification tasks for a synthetic dataset, as well as a dataset of brain vessel models of patients of various ages. The classifying model learns the optimal parameters p and T in order to, for example, classify patients by age based on their brain vessel structure. The parameter q is fixed equal to 1, the standard value, and T is the cumulative density of a mixed Gaussian model. The key observation is that, when p is large, then the model focuses on long bars, and when p is close to 1, then it focuses on short bars. It had been observed that short bars are more relevant for this dataset [5], and we recover this result.

Contributions

In this paper I was a main contributor to Section 3, which contains the fundamental theoretical results that allow for the results and computations in the rest of the paper.

CRediT classification. **Isaac Ren:** conceptualization (equal); formal analysis (lead); visualization (equal); writing – original draft preparation (equal); writing – review & editing (equal).



C Barcode matchings and p-norms of persistence modules

In this paper, we refine the tools introduced in Paper B. In particular, we extend the notion of bar-to-bar morphisms (Definition 4.3) to pointwise finite-dimensional persistence modules and prove analogous results to those of Paper B. In particular, we develop a simpler method of constructing bar-to-bar morphisms from mono- and epimorphisms. We present the results here for tame persistence modules; see Paper C for the full results for p.f.d. modules.

4.5 **Definition.** Let $V \cong \bigoplus_j \mathbf{k}_{[a_j,b_j)}$ and $W \cong \bigoplus_i \mathbf{k}_{[c_i,d_i)}$ be tame persistence modules with fixed persistence bases. For every morphism $f\colon V \to W$, we define $\Delta f\colon V \to W$ as the projection of f along the map of vector spaces

$$\begin{aligned} \hom(X,Y) &\cong \bigoplus_{i,j} \hom(\mathbf{k}_{[a_j,b_j)}, \mathbf{k}_{[c_i,d_i)}) \\ &\to \bigoplus_{\substack{i,j \\ b_j = d_i}} \hom(\mathbf{k}_{[a_j,b_j)}, \mathbf{k}_{[c_i,d_i)}) \\ &\hookrightarrow \hom(X,Y). \end{aligned}$$

In other words, considering a matrix representation of f, we only keep the nonzero coefficients whose row and column are labeled by bars with the same end-point. ■

4.6 **Theorem** (Theorem C.35). *For every morphism f in* **Tame***, the morphism Δf is bar-to-bar.*

We also prove that p-Wasserstein distances are pseudometrics for p.f.d. persistence modules using the noise system approach. Our approach to proving the inequality of p-norms of cokernels uses the integral of the rank function. In particular, by Proposition C.59, for all $p \in (1,\infty)$,

$$\|V\|_p^p = \int_{\Omega(\mathbf{R})} p(p-1)(t-s)^{p-2} \operatorname{rank} V(s \le t) \mathrm{d}s\mathrm{d}t.$$

In this more general setting, we also prove the isometry theorem between the algebraic and combinatorial Wasserstein distances.

4.7 **Theorem** (Theorem C.68). *For all $p \in [1,\infty]$ and all V and W in* **Pfd***, we have*

$$d_{\mathscr{S}^p}(V,W) = d_p(\mathscr{B}(V), \mathscr{B}(W)).$$

In addition to this new proof of the isometry result, we also study the properties of matchings induced by morphisms. Note that a bar-to-bar morphism can be seen as an algebraic representation of a barcode matching: by Definition 4.3, such a morphism has a matrix representation with at most one nonzero coefficient in each row and column. The set of coordinates of nonzero coefficients defines a matching between the persistence bases, and therefore the barcodes, of the domain and codomain. We thus define matchings induced by morphisms. Building on the results of [3, 29], we show that our matchings are **additive** but not **functorial** (Definition C.38). In fact, we show that an operator that associates to each morphism a matching cannot be both additive and functorial (Proposition C.40).

Using the strategy of [3], we define the induced matching of a morphism $f\colon V \to W$ by taking an epi-mono factorization $f = m \circ e$, inducing matchings on m and e, and then composing them together. In particular, there exists an algorithm that computes the epi-mono factorization such that the resulting matching $\mu(f)$ is that of [29] (Section C.3.4).

Contributions

I was an equal contributor throughout the conceptualization and writing of the paper.

CRediT classification. **Isaac Ren:** conceptualization (equal); formal analysis (lead); writing – original draft preparation (equal); writing – review & editing (equal).

D Morse theory of Euclidean distance functions from algebraic hypersurfaces

In this paper we generalize Morse theory to distance functions $f := \mathrm{dist}_Y|_X$ from a subset $Y \subseteq \mathbf{R}^n$ restricted to a subset $X \subseteq \mathbf{R}^n$. We show that, under certain conditions, the distance function f is a continuous selection:

4.8 **Proposition** (Proposition D.36). *Suppose that Y is smooth and let $x \in \mathbf{R}^n$ be a regular value of the exponential map $\varphi\colon NY \to \mathbf{R}^n$, where NY denotes the normal bundle of Y*[1]*. Then*

[1]At every point y in Y, the normal space N_yY is the orthogonal space to the tangent space

1. *$B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set $\{y_0, \dots, y_k\}$,*

and for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

2. *for every $i \in \{0, \dots, k\}$, the function $f_i := \mathrm{dist}_{B(y_i, \varepsilon) \cap Y}$ is $\mathscr{C}^2$ on $B(x, \delta)$;*
3. *the function $\mathrm{dist}_Y|_{B(x,\delta)}$ is a continuous selection of the f_i: specifically, $\mathrm{dist}_Y|_{B(x,\delta)} = \min_i f_i$.*

We can therefore apply the results of [1] to distance functions $\mathrm{dist}_Y|_X$ where all critical points are regular points of the exponential map on Y. Recall from Definition 3.7 that a nondegenerate critical point x of a continuous selection has two indices, a piecewise linear index $k(x)$ and a quadratic index $\iota(x)$. Given a pair (X, Y) of subsets of $\mathbf{R}^n$ such that $\mathrm{dist}_Y|_X$ only has nondegenerate critical points, we denote by $C_{k,\iota}(X, Y)$ the critical points with indices (k, ι). We recover the Morse lemmas and inequalities presented in Section 3.1. Moreover, we can exchange the roles of X and Y: if $\mathrm{dist}_X|_Y$ also only has nondegenerate critical points, then combining the resulting Morse inequalities gives the following equality relating the Euler characteristics of X and Y and the number of critical points (Corollary D.44):

$$\chi(Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(X, Y) = \chi(X) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \# C_{k,\iota}(Y, X).$$

In the second half of the paper, we study the case where X and Y are generic algebraic hypersurfaces, i.e., zero sets of polynomial functions. By **generic** we mean that the results hold for a dense open subset of polynomials. We summarize our results as follows:

4.9 **Theorem.** *Let $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$ be generic polynomials, and let $X = Z(p)$ and $Y = Z(q)$ be their zero sets. Then, depending on the values of $d_1 = \deg(X)$ and $d_2 = \deg(Y)$ the following is true:*

1. *if $d_1 \geq 3$ and $d_2 \geq 4$, then the function* dist *is a continuous selection near each of its critical points (Corollary D.63);*
2. *if $d_1, d_2 \geq 4$, then every critical point of* dist *not in Y is nondegenerate (Theorem D.69);*

$T_y Y$.

3. *if* $d_1, d_2 \geq 3$*, then for every* $k \in \{1, \ldots, n\}$*, the number of* k*-critical points of* $\operatorname{dist}_Y|_X$ *can be bounded by*

$$\#C_{k,*}(X,Y) \leq c(k,n) \deg(X)^n \deg(Y)^{n(k+1)},$$

for some number $c(k,n) > 0$ *depending on* n *and* k *only (Theorem D.60).*

We get analogous results when $X = \mathbf{R}^n$ (Remark D.70).

Contributions

In this paper I was the main contributor to Section 4, which applies the theory of the previous sections to algebraic hypersurfaces, and I contributed to Section 2 with the proofs and example figures.

CRediT classification. **Isaac Ren:** conceptualization (equal); formal analysis (lead); visualization (lead); writing – original draft preparation (equal); writing – review & editing (equal).

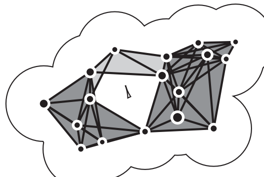
E Nondegeneracy of distance functions between complete intersections

This paper is a direct generalization of Paper D's results on algebraic hypersurfaces to complete intersections. There are no surprises when extending the results, but the computations require some extra care.

Contributions

I was the sole contributor to this paper.

CRediT classification. **Isaac Ren:** conceptualization (lead); formal analysis (lead); writing – original draft preparation (lead); writing – review & editing (lead).

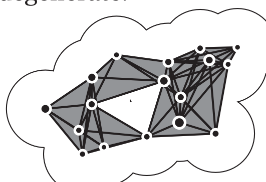


5 Conclusions and outlook

It was evening all afternoon.
It was snowing
And it was going to snow.
The blackbird sat
In the cedar-limbs.

In this thesis, we explored several ways in which algebraic invariants for filtered spaces can be defined and computed. On the one hand, we contributed to the theory of **persistent algebra**, a collection of research efforts applying linear and homological algebra to the study of persistence modules. Paper A develops relative homological algebra for general poset-indexed persistence modules and demonstrates the computational power of local Koszul complexes in the computation of relative Betti diagrams. Papers B and C use linear algebra, and in particular matrix representations of morphisms, to pass from algebraic persistence modules to combinatorial barcode matchings.

On the other hand, we contributed to the emerging field of **metric algebraic geometry** with our study of Morse theory for distance functions between algebraic varieties in Papers D and E. In particular, we identify two indices for nondegenerate critical points, which encode the topological and algebraic properties of filtered spaces, and we show that generic distance functions between algebraic varieties are "Morse," i.e., their critical points are all nondegenerate.



Both persistent algebra and metric algebraic geometry are open and actively developing fields. As we improve our understanding in these fields, we will obtain broader applications in mathematics and other sciences, as well as more effective computational methods. We now present some ongoing work and interesting partial results in the study of persistence modules and algebraic distance functions. We freely use the notations and results of the included papers.

5.1 Replacing kernels with cokernels

Following discussions with Andrea Guidolin and Bjørnar Gullikstad Hem.

Recall from Example F of Paper A that for the lower hook parameterization, we have $\mathrm{Nat}(\mathscr{T}(a,b),F) = \ker F(a \le b)$. Denote by π_1 and π_2 the two projections $\Omega(I) \to I$. We get two functors $F\pi_1$ and $F\pi_2$ and a natural morphism $\alpha \colon F\pi_1 \to F\pi_2$ between them. The long exact sequence in $\mathrm{Fun}(\Omega(I), \mathbf{Vect})$

$$0 \to \ker F(- \le -) \hookrightarrow F\pi_1 \xrightarrow{\alpha} F\pi_2 \twoheadrightarrow \operatorname{coker} F(- \le -) \to 0$$

induces the following short exact sequences:

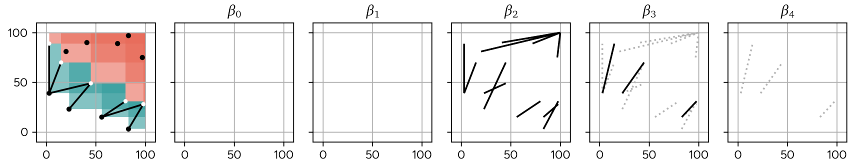
$$\begin{gathered} 0 \to \ker F(- \le -) \hookrightarrow F\pi_1 \twoheadrightarrow \operatorname{im} \alpha \to 0, \\ 0 \to \operatorname{im} \alpha \hookrightarrow F\pi_2 \twoheadrightarrow \operatorname{coker} F(- \le -) \to 0. \end{gathered}$$

Now suppose that I is a two-dimensional grid (this should generalize to higher dimensions). Let $(a \le b)$ be an element of $\Omega(I)$ with at least 3 parents: in other words, a and b share at most one coordinate (of the two). Then we can show that the Koszul complex of $F\pi_i$, for $i = 1, 2$, is a product of chain complexes, one of which is contractible, and thus the homology of the Koszul complex is 0 in all degrees.

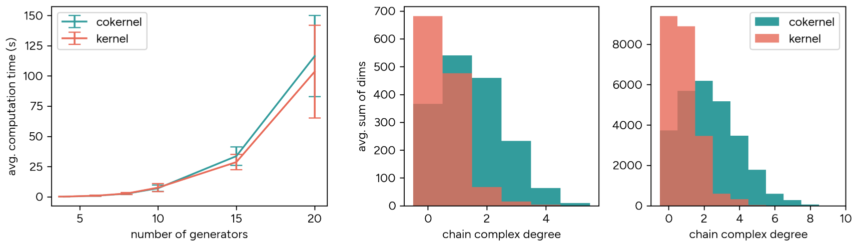
Short exact sequences induce long exact sequences in homology: using the 0's of the homology of $F\pi_i$, we get the following isomorphisms for all $d \ge 0$:

$$H_d(\ker F(- \le -)) \cong H_{d+1}(\operatorname{im} \alpha) \cong H_{d+2}(\operatorname{coker} F(- \le -))$$

Therefore, in order to compute Betti diagrams relative to lower hooks, we can study the Koszul complex of $\operatorname{coker} F(- \le -)$ instead of $\ker F(- \le -)$. This has



(a) Result of computing the homology of the cokernel complex; compare with Figure 4.1.



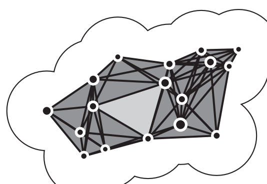
(b) Average (of 5 runs) time to compute relative Betti diagrams with respect to the number of generators.

(c) Average (of 10 runs) sum of dimensions per degree in the Koszul complex of kernels and cokernels for randomly generated persistence modules with 5 (left) and 10 (right) generators.

Figure 5.1: Results of the cokernel approach to computing Betti diagrams relative to lower hooks.

some theoretical computational advantages: in order to compute a kernel basis, we need $O(n^2)$ space, where n is the ambient dimension. In order to compute a cokernel basis, we only need $O(n)$ space, because we can choose representatives in the quotient space that correspond to canonical basis elements.

I have implemented this cokernel approach in Python, to middling success: see Figure 5.1 for some experimental results. In Figure 5.1a, we observe the exact same relative Betti diagrams as in Figure 4.1, except translated by 2 degrees. In Figure 5.1b, we see that the time required to compute relative Betti diagrams does not change when switching to cokernels, and in fact worsens a little bit. This may be explained by the fact that the cokernel complexes are larger in general than the corresponding kernel complexes: see Figure 5.1c. It is unclear where this is by nature or by the synthetic way in which they were constructed for this experiment, and this is worth exploring further.



5.2 Space of rank functions

With Wojiech Chachólski, Andrea Guidolin, Christina Kapatsori, Woojin Kim, Francesca Tombari.

The rank function $\operatorname{rank} V$ of a persistence module V over a poset I (Definition 2.38) satisfies the following inequalities. For all chains $a \le b \le c \le d$ in I,

1. (**monotonicity**) $\operatorname{rank} V(a \le d) \le \operatorname{rank} V(b \le c)$, and
2. (**chain-supermodularity**) $\operatorname{rank} V(a \le d) + \operatorname{rank} V(b \le c) \le \operatorname{rank} V(a \le c) + \operatorname{rank} V(b \le d)$.

The name **chain-supermodularity** comes from **supermodularity**, a notion studied in probability and order theory [43]. Monotone, chain-supermodular functions form a positive convex cone. The broad aim of this project is to study this cone and investigate if we can express rank functions as convex combinations of extremal rays.

As a first step, we study **chain-modular** functions, which satisfy the chain-supermodularity condition but with an equality instead of inequality. An interesting question is: which functors have rank functions that are chain-modular? This is ongoing work.

5.3 Rank-based noise systems for multiparameter persistence

With Håvard Bakke Bjerkevik, Andrea Guidolin, and Martina Scolamiero.

In Paper C, we show that the p-norm for a one-dimensional persistence module can be computed using the integral of its rank function (Proposition C.59). Here, we generalize this to multiparameter persistence. For $p > 1$ and $V\colon \mathbf{R}^n \to \mathbf{Vect}$ an n-dimensional persistence module, consider the following integral:

$$\|V\|_p := \left(\int_{\vec{x} \le \vec{y} \in \Omega} p(p-1)(y_1 - x_1)^{p-n-1} \operatorname{rank} V(\vec{x} \le \vec{y}) \right)^{\frac{1}{p}},$$

where $\Omega := \{(\vec{x}, \vec{y}) \in (\mathbf{R}^n)^2 \mid \vec{x} \le \vec{y}\}$ for the product order on $\mathbf{R}^n$. We have the following result:

5.1 **Proposition.** *Let $p \in (1, \infty)$. The p-norm on n-dimensional persistence modules defines in amplitude. In other words, the p-norm defines a noise system.*

Sketch of proof. The first short exact sequence axiom (see Definition 2.57) comes from Proposition C.65, as in the one-dimensional case.

To prove the second, harder short exact sequence axiom, we proceed as follows. We restrict the rank function of V to lines in $\mathbf{R}^n$ parameterized as $\ell\colon t \mapsto t\vec{a} + \vec{b}$, where $\vec{a} = (1, a_2, \dots, a_n)$ and $\vec{b} = (0, b_2, \dots, b_n)$. Then we parameterize our pairs of points $(\vec{x}, \vec{y})$ in Ω by scalars s and t in $\mathbf{R}$:

$$\vec{x} = s\vec{a} + \vec{b}, \qquad \vec{y} = t\vec{a} + \vec{b}.$$

Then we have the change of parameters

$$\begin{aligned} \|V\|_p^p &= \int_{(\vec{x}\le\vec{y})\in\Omega} p(p-1)(y_1 - x_1)^{p-n-1} \operatorname{rank} V(\vec{x} \le \vec{y}) \\ &= \int_{\vec{a}\ge 0} \int_{\vec{b}\in\mathbf{R}^n} \int_{s=-\infty}^{+\infty} \int_{t=s}^{+\infty} p(p-1)(t-s)^{p-2} \operatorname{rank} V(\vec{p}, \vec{q}) \mathrm{d}t\mathrm{d}s\mathrm{d}\vec{b}\mathrm{d}\vec{a}, \end{aligned} \tag{5.1}$$

where the Jacobian of the change of parameters is

$$J_\varphi = \begin{array}{c} \\ p_1 \\ q_1 \\ p_2 \\ q_2 \\ \vdots \\ p_n \\ q_n \end{array} \begin{array}{c} \begin{array}{cccccccc} s & t & a_2 & b_2 & \cdots & a_n & b_n \end{array} \\ \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ a_2 & 0 & s & 1 & & & \\ 0 & a_2 & t & 1 & & & \\ \vdots & \vdots & & & \ddots & & \\ a_n & 0 & & & & s & 1 \\ 0 & a_n & & & & t & 1 \end{pmatrix} \end{array}$$

In particular, we have $|\det J_\varphi| = (t-s)^{n-1}$.

For a given line $\ell\colon t \mapsto t\vec{a} + \vec{b}$, we apply the result that the p-norm gives a noise system in the one-dimensional case. In particular, given a short exact sequence $0 \to U \to V \to W \to 0$ of n-dimensional persistence modules, we use the characterization of p-norms as integrals of the rank function Proposition C.59 to obtain the following triangular inequality:

$$\begin{aligned} &\left(\int_{s=-\infty}^{+\infty} \int_{t=s}^{+\infty} p(p-1)(t-s)^{p-2} \operatorname{rank} V(\vec{x} \le \vec{y}) \mathrm{d}t\mathrm{d}s \right)^{\frac{1}{p}} \\ &\le \left(\int_{s=-\infty}^{+\infty} \int_{t=s}^{+\infty} p(p-1)(t-s)^{p-2} \operatorname{rank} U(\vec{x} \le \vec{y}) \mathrm{d}t\mathrm{d}s \right)^{\frac{1}{p}} \end{aligned}$$

$$+\left(\int_{s=-\infty}^{+\infty}\int_{t=s}^{+\infty} p(p-1)(t-s)^{p-2}\operatorname{rank} W(\vec{x}\leq\vec{y})\mathrm{d}t\mathrm{d}s\right)^{\frac{1}{p}}.$$

Let
$$f(\vec{a},\vec{b}) := (\int_{-\infty}^{\infty}\int_{s}^{\infty} p(p-1)(t-s)^{p-2}\operatorname{rank} U(\vec{x}\leq\vec{y})dtds)^{\frac{1}{p}},$$

and define g and h similarly. Then the previous equations rewrites as $f \leq g+h$.

We can apply Minkowski's inequality, which states that $(\int(f+g)^p)^{\frac{1}{p}} \leq (\int f^p)^{\frac{1}{p}} + (\int g^p)^{\frac{1}{p}}$ holds for $1 \leq p < \infty$. We get

$$\left(\int_{\vec{a}}\int_{\vec{b}} f(\vec{a},\vec{b})^p d\vec{b}d\vec{a}\right)^{\frac{1}{p}} \leq \left(\int_{\vec{a}}\int_{\vec{b}} g(\vec{a},\vec{b})^p d\vec{b}d\vec{a}\right)^{\frac{1}{p}} + \left(\int_{\vec{a}}\int_{\vec{b}} h(\vec{a},\vec{b})^p d\vec{b}d\vec{a}\right)^{\frac{1}{p}}.$$

By (5.1), this exactly says that

$$||V||_p \leq ||U||_p + ||W||_p,$$

and this is the hard axiom of noise systems. □

From here, we can study the pseudometric induced by this noise system, including its computation and applications. We can also compare with existing notions of ℓ^p-distances on multiparameter persistence modules [7].

5.4 p-norms of extensions

This is the content of Hugo Åkesson's Masters thesis [53], which I co-supervised with Wojciech Chachólski.

Let V and W be persistence modules. An **extension of** V **by** W is a persistence module E that fits into a short exact sequence

$$0 \to W \to E \to V \to 0.$$

We denote by $\operatorname{Ext}(V,W)$ the collection of extensions up to isomorphism (a morphism of extensions is one that commutes with the maps from W and to V), and it turns out that this collection forms a vector space. The question that Åkesson studied is: what is the maximal p-norm of an extension? We have results if W is a single bar:

5.2 **Theorem** ([53, Theorem 3.1]). *Let V be a tame persistence module and $[a,b)$ an interval of $\mathbf{R}$. Write*

$$F^{ab}(V) := \{(c,d) \in \mathscr{D}(V) \mid a < c \leq b < d\}.$$

Then there exists explicitly constructed bijection

$$\psi\colon \{\textit{antichains in } F^a b(V)\} \cong \mathscr{B}(\mathrm{Ext}(\mathbf{k}_{[a,b)}, V)),$$

where antichains are defined in [Paper A, §4.6] and $\mathscr{B}(-)$ is the set of barcodes of elements of the argument.

Moreover, we can equip antichains with a partial order $\preceq$ (see again [Paper A, §4.6]) such that the following result holds.

5.3 **Theorem** ([53, Theorem 3.2]). *We use the notations of Theorem 5.2 and fix $p \in [1,\infty]$. The function $A \mapsto \|\psi(A)\|_p$ from antichains in $F^{ab}(V)$ to p-norms is monotone. In other words, for antichains A and B in F^{ab},*

$$A \preceq B \quad \Rightarrow \quad \|\psi(A)\|_p \leq \|\psi(B)\|_p.$$

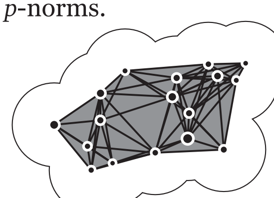
Notably, the poset of antichains of F^{ab} has a greatest element, the antichain of maximal elements of F^{ab}. By Theorem 5.3, the bijection ψ sends this antichain to the extension barcode with the greatest p-norm.

When W is arbitrary, Åkesson provides a recursive formula [53, Theorem 4.1]: if $W = W_1 \oplus W_2$ with $\mathrm{Ext}(W_1, W_2) = 0$ (this is always possible), then

$$\mathscr{B}(\mathrm{Ext}(W,V)) = \bigcup_{B \in \mathscr{B}(\mathrm{Ext}(W_2,V))} \mathscr{B}(\mathrm{Ext}(W_1, \mathscr{V}(B))), \tag{5.2}$$

where $\mathscr{V}(B)$ is a persistence module with barcode B. This gives an iterative strategy for computing $\mathscr{B}(\mathrm{Ext}(W,V))$: sort the bars of W compatibly with the product order on the start- and end-points, set W_2 equal to the first bar and W_1 to the sum of the rest: by construction, we get $\mathrm{Ext}(W_1, W_2) = 0$.

It is still an open problem to determine the extension with largest p-norm. This is useful, for example, in the study of the noise system axioms for p-norms.



5.5 Pair degree

Following discussions with Emil Horobeț.

In Papers D and E, we defined notions of algebraic critical points, and bounded their number. The number of algebraic critical points could define a new numerical invariant that generalizes the Euclidean distance degree [23] and the bottleneck degree [21]. Here we sketch its definition:

5.4 **Definition.** Let X and Y be real complete intersections defined by polynomials $\vec{p}$ and $\vec{q}$, respectively. We define the **pair degree of** X **and** Y to be the cardinality of the set $C_1(\vec{p}, \vec{q})$ of algebraic 1-critical points of the distance function $\mathrm{dist}_Y|_X$ (see Definition E.19). ■

When X is the entire space $\mathbf{R}^n$, this is the Euclidean distance degree, and when X is equal to Y, this is the bottleneck degree. Future works includes numerically computing the pair degree on examples, studying properties such as stability with respect to perturbations of X and Y, and studying algebraic k-critical points with $k \geq 2$.

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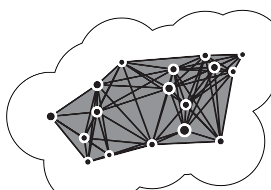

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Papers

Paper A

Koszul complexes and relative homological algebra of functors over posets

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Koszul Complexes and Relative Homological Algebra of Functors Over Posets

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Abstract

Under certain conditions, Koszul complexes can be used to calculate relative Betti diagrams of vector space-valued functors indexed by a poset, without the explicit computation of global minimal relative resolutions. In relative homological algebra of such functors, free functors are replaced by an arbitrary family of functors. Relative Betti diagrams encode the multiplicities of these functors in minimal relative resolutions. In this article we provide conditions under which grading the chosen family of functors leads to explicit Koszul complexes whose homology dimensions are the relative Betti diagrams, thus giving a scheme for the computation of these numerical descriptors.

Keywords Relative homological algebra · Poset representations · Betti diagrams · Koszul complexes · Topological data analysis · Multi-parameter persistent homology

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1 Introduction

1.1 Relative Homological Algebra and Koszul Complexes

Recently, there has been intense research activity [1, 5, 6] relating topological data analysis (TDA) with relative homological algebra [3, 4, 10, 11, 13] of finite dimensional vector space representations of finite posets. One way of thinking about such representations is as modules over certain finite dimensional algebras, allowing to phrase homological properties of modules in terms of homological properties of the algebra. This is also a common way for expressing relative homological properties of representations. In this case, endomorphism algebras of some chosen modules are often utilized (see e.g. [3, 5]). We however take another perspective, central in TDA, and study representations as functors. Local Koszul complexes [9] (see also [18, Sect. 6] and [20, Sect. 1]) are the key reasons why viewing representations as functors is particularly convenient in TDA. In favourable situations, these complexes can be used for calculations of Betti diagrams without explicitly constructing global minimal resolutions, especially if we have some control over the sets of parents of elements in the indexing poset.

The aim of this article is to show that Koszul complexes can also be used to calculate relative Betti diagrams. To do this, we translate from a relative homological algebra of functors indexed by a given poset to the standard homological algebra of functors indexed by a new poset. We are interested in translations for which the homology of the Koszul complexes over the new poset computes the relative Betti diagrams in a way that can be implemented in software (see for example [23]). One of our main contributions in this article is a translation construction which allows us to characterize the set of poset elements (called the degeneracy locus) where local Koszul complexes fail to calculate the relative Betti diagrams. Importantly, in all examples that are currently relevant for TDA the degeneracy loci are negligible.

Aiming at a self-contained presentation, and believing that the tools we introduce may be of interest to a broader audience, we favor an explicit classical formulation of relative homological algebra [10].

1.2 Relative Homological Approximations

One way of understanding an object is by approximating it. In this article we focus on approximations coming from relative homological algebra (see [13]), where arbitrary objects in an abelian category are approximated by finite direct sums of objects of a chosen collection $\mathcal{P}$. For example, one may ask whether a given object M is isomorphic to a finite direct sum of elements in $\mathcal{P}$. To answer this question we look at the category $\bigoplus\mathcal{P}\downarrow_M$ of morphisms from finite direct sums of elements in $\mathcal{P}$ into M. We are interested in situations where this category has a certain distinguished object $C_0 \to M$: for example, a terminal object. However, cases where the category $\bigoplus\mathcal{P}\downarrow_M$ has a terminal object are not that common. More common is the existence of $C_0 \to M$ in the category $\bigoplus \mathcal{P} \downarrow_M$ satisfying the following two conditions. The first condition requires that every object in $\bigoplus \mathcal{P} \downarrow_M$ maps to $C_0 \to M$, in which case $C_0 \to M$ is called a $\mathcal{P}$-**epimorphism** (see 2.2). This of course holds if $C_0 \to M$ is terminal in $\bigoplus \mathcal{P} \downarrow_M$, where not only the existence but also the uniqueness of morphisms into $C_0 \to M$ is required. The second condition substitutes this uniqueness by a minimality condition: every endomorphism of $C_0 \to M$ is an isomorphism. A morphism $C_0 \to M$ satisfying these two conditions (being a minimal $\mathcal{P}$-epimorphism) is called a **minimal** $\mathcal{P}$**-cover** of M (see 2.4). If it exists, then the minimal $\mathcal{P}$-cover of M is unique up to isomorphism. We regard the minimal $\mathcal{P}$-cover of M as an approximation of M by finite direct sums of elements in $\mathcal{P}$. In particular, M is isomorphic to such a direct sum if, and only if, its minimal $\mathcal{P}$-cover is an isomorphism.

If finite direct sums of elements in $\mathcal{P}$ are uniquely determined by the multiplicities of their summands, then we say that $\mathcal{P}$ is **independent** (see 2.1). In this case, there is a unique function $\beta^0_{\mathcal{P}}M : \mathcal{P} \to \mathbb{N}$ whose value at A in $\mathcal{P}$ is the multiplicity of A in the minimal $\mathcal{P}$-cover C_0 of M. The function $\beta^0_{\mathcal{P}}M$ is called the 0^{th} $\mathcal{P}$**-Betti diagram** of M and is a numerical descriptor of the approximation of M by finite direct sums of elements in $\mathcal{P}$. This is only the 0^{th} step. We can continue by considering the minimal $\mathcal{P}$-cover of the kernel of $C_0 \to M$. By doing this inductively we obtain a chain complex $\cdots \to C_n \to \cdots \to C_0 \to M$ called a **minimal** $\mathcal{P}$**-resolution** of M. Each object C_d in this complex is described by a function $\beta^d_{\mathcal{P}}M : \mathcal{P} \to \mathbb{N}$ called the d^{th} $\mathcal{P}$**-Betti diagram of** M, where, for all A in $\mathcal{P}$, the number $\beta^d_{\mathcal{P}}M(A)$ is the multiplicity of A in C_d (see 2.6).

Thus every independent collection $\mathcal{P}$, for which all objects admit minimal $\mathcal{P}$-covers, leads to numerical invariants in the form of $\mathcal{P}$-Betti diagrams. The focus of this article is to provide a method of constructing such collections $\mathcal{P}$ and to discuss a strategy to effectively compute the associated Betti diagrams in the category $\mathrm{Fun}(I, \mathrm{vect}_K)$ of functors indexed by a finite poset I with values in finite dimensional K-vector spaces.

1.3 Related Work

Since vector space valued functors indexed by posets have played an important role across many areas of mathematics, their categories are well studied, and in particular their homological properties are well understood: see for example [3] and references therein, where such functors are interpreted as modules over finite dimensional algebras. Recently there has also been a growing interest in independent collections in $\mathrm{Fun}(I, \mathrm{vect}_K)$ [2, 5, 6] and associated numerical invariants coming from the TDA community, where such functors are commonly called persistence modules, and the poset I is usually $\mathbb{R}^n$ with the product order. In particular, there are several publications and implementations of various algorithms to calculate numerical homological invariants of such functors (e.g. classical Betti diagrams, rank functions, generalized Euler characteristics) and using them in applications, see for example [15–17, 21, 22, 24].

The idea of approximating persistence modules by objects in a chosen collection appears in [1, 2, 5, 6, 14]. In [14], without relying on homological algebra, approximations in terms of the intervals in the poset are shown to yield a generalization of rank functions, an invariant of persistence modules. In [1], the authors approximate persistence modules in a Grothendieck group generated by interval modules (called spread modules in [5] and in our work, see Example D) using quiver representation theory without resorting to homological algebra methods. Decompositions of rank functions into linear combinations of rank functions of simpler modules, such as rectangles and lower hooks, are explained in [6]. Importantly, one such decomposition (called minimal rank decomposition) is defined via relative homological algebra of persistence modules with respect to the collection of lower hooks, which we consider in Example F. In [2], relative homological algebra with respect to interval modules is studied using the language of representations of finite dimensional algebras. For homological algebra results for persistence modules over more general posets, we can also look at [8] and [19]. In the former, the authors prove various standard homological algebra results for persistence modules, with a focus on the one-parameter case (where the poset is the real line). In the latter, the author studies presentations of persistence modules, with focus on infinite posets, and concludes with a Hilbert syzygy-type theorem. The authors of [5] present a way of obtaining relative projective resolutions from standard projective resolutions by considering modules over the ring $\mathrm{End}(\bigoplus \mathcal{P})^{\mathrm{op}}$ and finding a resolution of $\hom(\bigoplus \mathcal{P}, M)$. This gives an automatic correspondence of exact structures, but there is no reason for the standard projective resolution of $\hom(\bigoplus \mathcal{P}, M)$ to be more easily computed than the relative projective resolution of M.

Our original motivation was to find a homological interpretation of the approximations presented in these articles that would allow for explicit calculations. This article builds on the realization that local Koszul complexes can be used for this purpose. Koszul complexes of modules are well studied, see e.g. [18, Sect. 6]. Koszul complexes of graded modules, as presented in [20], have a direct interpretation in terms of persistence modules, based on the equivalence between the category of n-graded modules over the polynomial ring $S{:=}K[x_1, \ldots, x_n]$ and the category $\mathrm{Fun}(\mathbb{N}^n, \mathrm{vect}_K)$, where $\mathbb{N}^n$ is endowed with the product order. Here we sketch this connection, referring to [12, Sect. 3] for more details. The Koszul complex $\mathcal{K}$ as defined for example in [20, Def. 1.26] is a minimal free resolution of the field K in the category of n-graded S-modules [20, Prop. 1.28]. Given an n-graded S-module M, the Koszul complex of M at $a \in \mathbb{N}^n$ is the grade a part of the n-graded chain complex $M \otimes_S \mathcal{K}$. Its d^{th} homology module is the grade a part of the n-graded S-module $\mathrm{Tor}_i^S(M, K)$, whose dimension is the d^{th} Betti diagram (with respect to the standard projectives) of M at a. This value of the Betti diagram can be computed, for a grade $a \in \mathbb{N}^n$, by looking only at M restricted to the grades in a small subposet of $\mathbb{N}^n$ isomorphic to $\{0 < 1\}^k$ for some $k \leq n$, whose elements are a together with its parents and all their meets. In this work, we study local Koszul complexes of functors indexed by more general finite posets (see 3.7), more details on which can be found in [9], and use them in the context of relative homological algebra.

The above approach starts with a global resolution of K and considers local parts of $M \otimes_S \mathcal{K}$ at different grades. Such a global resolution however may not have an explicit formula, which typically would require global regularity assumptions on the indexing poset. Requiring local regularity only at some grades is a much weaker assumption.

In such grades we have an explicit Koszul complex construction which we exploit. For that purpose, the functorial perspective is more advantageous than the language of representations of finite dimensional algebras used in [2, 5, 6]. Our aim is to use Koszul complexes not only as a theoretical tool, but as a computable construction to determine relative Betti diagrams.

1.4 Koszul Complexes and Thinness

Homological algebra characterizes relevant objects by universal properties, regardless of the category to which they belong, instead of by concrete constructions which may vary greatly with the category. This offers a powerful language in which to express and solve mathematical problems. However, extracting calculable invariants, which is essential in TDA, from universally defined objects is often difficult, costly, and sometimes not even feasible. Therefore the challenge is to introduce a version of homological algebra which retains some conceptual power while allowing for some of the relevant invariants to be described explicitly and algorithmically.

We strike a balance between mathematical and computational effectiveness by using a strategy built on the following two observations. The first is related to the standard independent collection $\mathcal{S} = \{K(a, -) \mid a \in I\}$ consisting of all free functors on one generator (see beginning of Sect. 3). This collection is always independent and, when I is finite, every functor in $\mathrm{Fun}(I, \mathrm{vect}_K)$ admits a minimal $\mathcal{S}$-resolution. The first key observation is that under additional assumptions, for example that I is an upper semilattice, for every element a in I and every functor $F\colon I \to \mathrm{vect}_K$, there exists an explicit chain complex $\mathcal{K}_a F$, called **Koszul complex** (see 3.7 and discussion in 1.3), whose homology in degree d has dimension equal to the value of the Betti diagram $\beta^d_{\mathcal{S}} F(K(a, -))$ (see Theorem 3.8 and [9, Section 10]). Because of its explicitness, the Koszul complex is an effective tool for computing the values of the Betti diagram with respect to the collection $\mathcal{S}$.

To have an analogous construction for a more general collection $\mathcal{P}$ in $\mathrm{Fun}(I, \mathrm{vect}_K)$, we require a grading of its elements. Thus, instead of an unstructured collection, our starting point is a functor $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ indexed by a finite poset $(J, \preccurlyeq)$. Such a functor leads to a collection $\mathcal{P} := \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$. This grading allows for a translation between functors indexed by I and functors indexed by J via a pair of standard adjoint functors (see 5.2 and 5.3):

$$\mathrm{Fun}(I, \mathrm{vect}_K) \underset{\mathcal{L}}{\overset{\mathcal{R}}{\rightleftarrows}} \mathrm{Fun}(J, \mathrm{vect}_K)$$

where $\mathcal{R} := \mathrm{Nat}_I(\mathcal{T}, -)$ assigns to $M\colon I \to \mathrm{vect}_K$ the functor $\mathrm{Nat}_I(\mathcal{T}(-), M) : J \to \mathrm{vect}_K$. The functor $\mathcal{T}$ is called **thin** if the unit of this adjunction $\eta_a\colon K(a, -) \to \mathcal{R}\mathcal{L}K(a, -)$ is an epimorphism for every a in J. The second key observation is recognizing the importance of the thinness assumption. This assumption enables us to translate between the homological properties of M in $\mathrm{Fun}(I, \mathrm{vect}_K)$ relative to $\mathcal{P}$ and the standard homological properties of $\mathcal{R}M$ in $\mathrm{Fun}(J, \mathrm{vect}_K)$.

1.5 Results

Let I and J be finite posets, and $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ a thin functor. We give a self-contained proof that the collection $\mathcal{P} = \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$ is independent (5.13). Moreover, we prove that the adjunction between $\mathcal{L}$ and $\mathcal{R}$ leads to a one-to-one correspondence between minimal covers: for every functor M in $\mathrm{Fun}(I, \mathrm{vect}_K)$, a natural transformation $C_0 \to \mathcal{R}M$ is a minimal cover in $\mathrm{Fun}(J, \mathrm{vect}_K)$ if, and only if, its adjoint $\mathcal{L}C_0 \to M$ is a minimal $\mathcal{P}$-cover in $\mathrm{Fun}(I, \mathrm{vect}_K)$ (Theorem 5.14). Although we show that every functor M in $\mathrm{Fun}(I, \mathrm{vect}_K)$ admits a minimal $\mathcal{P}$-resolution (Corollary 5.16(1)), this resolution cannot in general be obtained via the adjunction. This means that in general the $\mathcal{P}$-Betti diagrams of M differ from the standard Betti diagrams of $\mathcal{R}M$. Elements of J where this happens are called degenerate and form the degeneracy locus (5.18). If J is an upper semilattice, our main result (Corollary 5.20) identifies elements outside the degeneracy locus, for which we can use Koszul complexes to calculate the values of $\mathcal{P}$-Betti diagrams. From a computational perspective, we obtain Algorithm 1 for computing relative Betti diagrams, given a thin functor.

Algorithm 1: Betti diagrams relative to a parameterization $\mathcal{T}$

Input: I finite poset, J finite upper semilattice,
$\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ satisfying the conditions of Corollary 5.20,
$M\colon I \to \mathrm{vect}_K$ functor,
$a \in J$ poset element such that $\mathcal{T}(a) \neq 0$.
Output: the relative Betti diagrams of M at $\mathcal{T}(a)$ relative to $\{\mathcal{T}(b) \mid b \in J,\ \mathcal{T}(b) \neq 0\}$.
$\mathcal{U}(a) \leftarrow \{a_1, \dots, a_n\}$ parents of a; `// parents defined in 3.7`
$J_0 \leftarrow \{\varnothing\}$;
for $d = 1, \dots, n$ **do** $J_d \leftarrow \{S \subseteq \mathcal{U}(a) \mid S \text{ bounded below},\ |S| = d\}$ **for** $S \in \bigcup_{d=0}^{n} J_d$ **do**
$\mathcal{R}M(\bigwedge S) \leftarrow \mathrm{Nat}(\mathcal{T}(\bigwedge S), M)$; `//` $\bigwedge \varnothing = a$ `by convention`
for $d = 0, \dots, n-1$ **do**
  for $S \in J_d$**,** $S' \in J_{d+1}$**, and** $S \subset S'$ **do**
    Write $i_0 < \dots < i_d$ such that $S' = \{a_{i_0}, \dots, a_{i_d}\}$;
    $k_{S,S'} \leftarrow k$ such that $S' \setminus S = a_{i_k}$;
    $b \leftarrow \bigwedge S$; $c \leftarrow \bigwedge S'$;
    $\partial_{S,S'} \leftarrow \big(\mathrm{Nat}(\mathcal{T}(b > c), M)\colon \mathcal{R}M(c) \to \mathcal{R}M(b)\big)$;
  end
end
`// Construction of chain complex` $(\mathcal{K}, \partial)$ `from 3.7`
$\mathcal{K} \leftarrow \Big(\bigoplus_{S \in J_d} \mathcal{R}M(\bigwedge S)\Big)_{d=0,\dots,n}$;
$\partial \leftarrow \Big(\sum_{S \in J_d,\ S' \in J_{d+1}} (-1)^{k_{S,S'}} \partial_{S,S'}\Big)_{d=0,\dots,n-1}$;
$\beta \leftarrow \dim H_*(\mathcal{K}, \partial)$;
return β

Furthermore, we produce examples of thin functors $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$, and we apply our results to explicitly calculate the associated relative Betti diagrams. We observe how numerical invariants introduced in other works, such as the rank invariant over lower hooks of [6] (Example F) and the single-source homological spread invariant of [5] (Example E), are included in our general framework.

1.6 Organization

We begin in Sect. 2 by introducing the theory of relative homological algebra, including the notions of independence and acyclicity, covers and resolutions. Next, in Sect. 3 we recall standard homological algebra for functors indexed by finite posets. In particular, when the poset is an upper semilattice, we show how to compute Betti diagrams explicitly via Koszul complexes. In Sect. 4, we discuss in more depth resolutions and Betti diagrams of certain functors called filtrations. In particular, we study subfunctors, and discuss several important notions: supports of functors, upsets, and antichains.

Section 5 is the central part of this work, where we introduce gradings of the form $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$. We define the key properties of thinness and flatness, which will allow us to compute relative Betti diagrams over I from standard Betti diagrams over J.

Finally, we connect to other works in Sect. 6 by exhibiting how rectangles, lower hooks, and spread modules, among others, are handled in our framework.

2 Relative Homological Algebra

In this section we recall a setup for relative homological algebra in an abelian category $\mathcal{M}$. This subject, initiated by Hochschild [13], has been extensively developed in the context of the representation theory of finite dimensional algebras (see for example [3, 4]). The explicit and self-contained treatment we provide here has the advantage of not assuming familiarity of the reader with such theory. We fix a collection $\mathcal{P}$ of objects in $\mathcal{M}$. What follows are fundamental notions of homological algebra relative to $\mathcal{P}$.

2.1 Freeness An object in $\mathcal{M}$ is called $\mathcal{P}$-**free** if it is isomorphic to a finite direct sum of elements in $\mathcal{P}$. For example, the zero object is $\mathcal{P}$-free. The collection $\mathcal{P}$ is called **independent** if, for every $\mathcal{P}$-free object F, there is a unique function $\beta\colon \mathcal{P} \to \mathbb{N}$ whose **support** $\operatorname{supp}\beta := \{A \mid \beta(A) \neq 0\}$ is finite and for which F is isomorphic to $\bigoplus_{A\in\mathcal{P}} A^{\beta(A)}$. If $\mathcal{P}$ is independent, then all of its elements are nonzero, and if two of its elements are isomorphic, then they must be equal. For example, if $\mathcal{M}$ is a Krull-Schmidt category and $\mathcal{P}$ is a collection of indecomposables, then it is independent if and only if no two elements in $\mathcal{P}$ are isomorphic.

2.2 Exactness Two composable morphisms $f\colon M \to N$ and $g\colon N \to L$ in $\mathcal{M}$ are said to form a $\mathcal{P}$**-exact sequence** if the following sequence of abelian groups is exact for every A in $\mathcal{P}$:

$$\hom(A, M) \xrightarrow{\hom(A,f)} \hom(A, N) \xrightarrow{\hom(A,g)} \hom(A, L).$$

A sequence of composable morphisms $\cdots \to M_d \to M_{d-1} \to \cdots$ is called $\mathcal{P}$-**exact** if every two of its consecutive morphisms form a $\mathcal{P}$-exact sequence.

If the sequence $M \xrightarrow{f} N \to 0$ is $\mathcal{P}$-exact, then f is called a $\mathcal{P}$-**epimorphism**. A morphism $f\colon M \to N$ is a $\mathcal{P}$-epimorphism if, and only if, $\hom(A, f)\colon \hom(A, M) \to \hom(A, N)$ is surjective for every A in $\mathcal{P}$ (i.e. every $g\colon A \to N$ can be expressed as the composition of some morphism $A \to M$ with f).

2.3 Projectives An object C_0 in $\mathcal{M}$ is called $\mathcal{P}$-**projective** if, for every $f\colon M \to N$ and $g\colon N \to L$ that form a $\mathcal{P}$-exact sequence, the following is an exact sequence of abelian groups:

$$\hom(C_0, M) \xrightarrow{\hom(C_0, f)} \hom(C_0, N) \xrightarrow{\hom(C_0, g)} \hom(C_0, L).$$

Note that if M is $\mathcal{P}$-projective, then $f\colon M \to N$ and $g\colon N \to L$ form a $\mathcal{P}$-exact sequence if, and only if, the composition gf is null and the induced morphism $M \to \ker(g)$ is a $\mathcal{P}$-epimorphism. Every $\mathcal{P}$-free object is $\mathcal{P}$-projective. If every $\mathcal{P}$-projective is $\mathcal{P}$-free, then the collection $\mathcal{P}$ is called **acyclic**.

2.4 Covers A $\mathcal{P}$-**cover** of an object M is a $\mathcal{P}$-epimorphism $C_0 \to M$ where C_0 is $\mathcal{P}$-projective. We say that the category $\mathcal{M}$ **has enough** $\mathcal{P}$**-projectives** if every object has a $\mathcal{P}$-cover.

A morphism between a $\mathcal{P}$-cover $C_0 \to M$ and a $\mathcal{P}$-cover $D_0 \to M$ is a morphism $f\colon C_0 \to D_0$ in $\mathcal{M}$ for which the following diagram commutes:

$$\begin{tikzcd} C_0 \arrow[d, "f"'] \arrow[dr] & \\ D_0 \arrow[r] & M \end{tikzcd}$$

Such a morphism is an isomorphism if f is an isomorphism in $\mathcal{M}$. There is always a morphism between two $\mathcal{P}$-covers $C_0 \to M$ and $D_0 \to M$, since C_0 is $\mathcal{P}$-projective and $D_0 \to M$ is a $\mathcal{P}$-epimorphism.

A $\mathcal{P}$-cover $C_0 \to M$ is called **minimal** if all its endomorphisms are isomorphisms. Every two minimal $\mathcal{P}$-covers of M are isomorphic: if $C_0 \to M \leftarrow D_0$ are two minimal $\mathcal{P}$-covers, then in any sequence $C_0 \to D_0 \to C_0 \to D_0$ of morphisms of the $\mathcal{P}$-covers, the first and second ones compose into an isomorphism by minimality, as well as the second and third ones, which implies that all these morphisms are isomorphisms.

2.5 Resolutions A sequence of composable morphisms in $\mathcal{M}$ of the form $\cdots \to C_1 \to C_0 \to M \to 0$ is called a $\mathcal{P}$**-resolution** of M if it is $\mathcal{P}$-exact and, for every $d \geq 0$, the object C_d is $\mathcal{P}$-projective. Equivalently, the sequence $\cdots \to C_1 \to C_0 \to M \to 0$ is a $\mathcal{P}$-resolution of M if the following three conditions are satisfied: (a) for every $d \geq 0$, the object C_d is $\mathcal{P}$-projective; (b) the composition of every two consecutive morphisms is 0; and (c) for every $d \geq 0$, the induced morphism $C_d \to \ker(C_{d-1} \to C_{d-2})$ is a $\mathcal{P}$-epimorphism, where $C_{-2} = 0$ and $C_{-1} = M$.

A $\mathcal{P}$-resolution of M is also written as a morphism of chain complexes, by $C \to M$, where M denotes a chain complex concentrated in degree 0 and $C = (\cdots \to C_1 \to C_0)$. An object M for which there is a $\mathcal{P}$-resolution $C \to M$ is called $\mathcal{P}$-**resolvable**. If there are enough $\mathcal{P}$-projectives, then all objects are $\mathcal{P}$-resolvable.

A morphism from a $\mathcal{P}$-resolution $C \to M$ to a $\mathcal{P}$-resolution $D \to M$ is a sequence of morphisms $f_d : C_d \to D_d$ for $d \geq 0$ for which the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \arrow[dr] & \\
 & & & M \\
\cdots \arrow[r] & D_1 \arrow[r] & D_0 \arrow[ur] &
\end{tikzcd}$$

Such a morphism is an isomorphism if f_d is an isomorphism for every $d \geq 0$. As with $\mathcal{P}$-covers, there is always a morphism between every two $\mathcal{P}$-resolutions of M.

A $\mathcal{P}$-resolution $C \to M$ is called **minimal** if all of its endomorphisms are isomorphisms. A $\mathcal{P}$-resolution $C \to M$ is minimal if, and only if, for every $d \geq 0$, the morphism $C_d \to \ker(C_{d-1} \to C_{d-2})$ is a minimal $\mathcal{P}$-cover, where $C_{-1} = M$ and $C_{-2} = 0$. Thus, if every object has a minimal $\mathcal{P}$-cover, then every object has a minimal $\mathcal{P}$-resolution that can be constructed inductively by taking minimal $\mathcal{P}$-covers of successive kernels. Every two minimal $\mathcal{P}$-resolutions of M are isomorphic.

2.6 Betti diagrams Suppose $\mathcal{P}$ is independent (see 2.1) and acyclic (see 2.3). Let $C \to M$ be a minimal $\mathcal{P}$-resolution. By acyclicity of $\mathcal{P}$, for every $d \geq 0$, the object C_d is $\mathcal{P}$-free. Thus, by independence of $\mathcal{P}$, there is a unique function, denoted by $\beta^d_{\mathcal{P}} M : \mathcal{P} \to \mathbb{N}$ and called the d^{th} $\mathcal{P}$-**Betti diagram of** M, for which C_d is isomorphic to

$$\bigoplus_{A \in \mathcal{P}} A^{\beta^d_{\mathcal{P}} M(A)}$$

Since any two minimal $\mathcal{P}$-resolutions of M are isomorphic, the $\mathcal{P}$-Betti diagrams of M do not depend on the choice of a minimal $\mathcal{P}$-resolution, and are invariants of the isomorphism type of M. One should note however that $\mathcal{P}$-Betti diagrams are only defined for objects that have minimal $\mathcal{P}$-resolutions.

To calculate the values of the d^{th} $\mathcal{P}$-Betti diagram of M, a standard strategy is to develop tools to calculate the 0^{th} $\mathcal{P}$-Betti diagram and then use the fact that, for a minimal resolution $C \to M$, we have the following sequence of equalities:

$$\begin{aligned}
\beta^d_{\mathcal{P}} M &= \beta^{d-1}_{\mathcal{P}} \ker(C_0 \to M) = \beta^{d-2}_{\mathcal{P}} \ker(C_1 \to C_0) \\
&= \cdots = \beta^0_{\mathcal{P}} \ker(C_{d-1} \to C_{d-2}).
\end{aligned}$$

Building a minimal resolution is an inductive procedure involving constructing minimal $\mathcal{P}$-covers of successive kernels, which typically requires a description of the entire 0^{th} $\mathcal{P}$-Betti diagram of these kernels.

One approach to calculating Betti diagrams that avoids this costly inductive procedure is the construction, given an object M in $\mathcal{M}$ and an element A in $\mathcal{P}$, of a chain complex $V = (\cdots \to V_1 \to V_0)$ of vector spaces such that, for all $d \geq 0$, $\dim H_d(V) = \beta^d_{\mathcal{P}} M(A)$. Ideally, this construction is systematic, or even functorial in M. This is the case for standard homological algebra for functors indexed by finite upper semilattices, discussed in the next section, where such chain complexes are given by **Koszul complexes**.

3 Standard Homological Algebra

Fix a field K, choose a finite poset $(J, \preccurlyeq)$, and consider the abelian category $\mathrm{Fun}(J, \mathrm{vect}_K)$ whose objects are functors indexed by J with values in the category vect_K of finite dimensional vector spaces. Morphisms in $\mathrm{Fun}(J, \mathrm{vect}_K)$ are the natural transformations. For an element a in J, denote by $K(a, -) \colon J \to \mathrm{vect}_K$ the composition of the following functors, where the symbol set denotes the category of finite sets:

$$\begin{array}{ccccc} J & \xrightarrow{\hom_J(a,-)} & \mathrm{set} & \xrightarrow{\quad \text{free} \quad} & \mathrm{vect}_K \\ & & S & \longmapsto & \bigoplus_S K \end{array}$$

That is, $K(a, b) = K$ if $b \succcurlyeq a$ and 0 otherwise, and $K(a, b \preccurlyeq c) = \mathrm{id}_K$ if $b \succcurlyeq a$ and 0 otherwise. The functor $K(a, -)$ is called **free on one generator in degree** a.

In this section we are going to describe the homological algebra of $\mathrm{Fun}(J, \mathrm{vect}_K)$, as presented in Sect. 2, relative to the collection $\mathcal{S} := \{K(a, -) \mid a \in J\}$. This corresponds to standard homological algebra and all the following statements are well known: see for instance [3, 26].

3.1 Independence The collection $\mathcal{S}$ is independent since its elements are indecomposable (see 2.1). This can be also shown using radicals, as in [3]. Recall that the **radical** of a functor $F \colon J \to \mathrm{vect}_K$ is a subfunctor $\mathrm{rad}(F) \subseteq F$ given by $\mathrm{rad}(F)(a) = \sum_{s \prec a} \mathrm{im}(F(s \prec a))$ for a in J. The quotient functor $H_0 F := F/\mathrm{rad}(F)$ is **semisimple**: that is, for every $a \prec b$ in J, the morphism $H_0 F(a \prec b)$ is the zero morphism. For a and b in J, $(H_0 K(a, -))(b)$ is 1-dimensional if $a = b$ and 0 otherwise. Thus, if F is free and isomorphic to $\bigoplus_{K(b,-) \in \mathcal{S}} K(b, -)^{\beta(K(b,-))}$, then $\beta(K(b, -)) = \dim H_0 F(b)$ and hence the number $\beta(K(b, -))$ is uniquely determined by the isomorphism type of F.

3.2 Exactness By the Yoneda lemma, for every functor $F \colon J \to \mathrm{vect}_K$, the linear map $\mathrm{Nat}_J(K(a, -), F) \to F(a)$, which assigns to a natural transformation $\varphi \colon K(a, -) \to F$ the element $\varphi_a(a \preccurlyeq a)$ in $F(a)$, is a bijection. Consequently, morphisms $f \colon F \to G$ and $g \colon G \to H$ in $\mathrm{Fun}(J, \mathrm{vect}_K)$ form an $\mathcal{S}$-exact sequence if, and only if, for every a in J, the linear maps $f_a \colon F(a) \to G(a)$ and $g_a \colon G(a) \to H(a)$ form an exact sequence of vector spaces. Thus, being $\mathcal{S}$-exact is the same as being exact. In particular, $f \colon F \to G$ is an $\mathcal{S}$-epimorphism if, and only if, it is a standard categorical epimorphism, which means $f_a \colon F(a) \to G(a)$ is surjective for all a in J. To detect this we can again use the radical:

3.3 Lemma *Let J be a finite poset. A natural transformation $f: F \to G$ in* $\mathrm{Fun}(J, \mathrm{vect}_K)$ *is an epimorphism if, and only if, the composition of f with the quotient $G \to G/\mathrm{rad}(G) = H_0 G$ is an epimorphism.*

Proof The only if part of this equivalence is clear. Conversely, suppose that the composition is an epimorphism and consider the set of all a in J for which f_a is not a surjection. If this set is nonempty, then it contains a minimal element a (since J is finite). Minimality of a means that $\mathrm{rad}(f)_a : \mathrm{rad}(F)(a) \to \mathrm{rad}(G)(a)$ is surjective. This, combined with the surjectivity of the composition $F(a) \to G(a) \to H_0 G(a)$, implies that f_a is surjective, which contradicts the assumption. □

Since $\mathcal{S}$-epimorphisms are the categorical epimorphisms, $\mathcal{S}$-projectives are the standard projectives, and it turns out that all of them are ($\mathcal{S}$-)free.

3.4 Proposition *Let J be a finite poset. The collection $\mathcal{S}$ in* $\mathrm{Fun}(J, \mathrm{vect}_K)$ *is acyclic (see 2.3).*

Proof We again use the radical. Let F be projective in $\mathrm{Fun}(J, \mathrm{vect}_K)$. Consider $H_0 F$ and the free functor $C_0 := \bigoplus_{a \in J} K(a, -)^{\dim H_0 F(a)}$. Note that $H_0 C_0$ and $H_0 F$ are isomorphic. We then use projectiveness to construct the following commutative diagram where all of the vertical arrows represent the quotient natural transformations:

$$\begin{array}{ccccc} F & \longrightarrow & C_0 & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ H_0 F & \simeq & H_0 C_0 & \simeq & H_0 F \end{array}$$

The vertical natural transformations in this diagram are epimorphisms, so by Lemma 3.3 the horizontal natural transformations are as well. Since the values of the functors are finite dimensional vector spaces, the horizontal natural transformations must therefore be isomorphisms. □

3.5 Minimal covers, resolutions, and Betti diagrams Let $F: J \to \mathrm{vect}_K$ be a functor. As in 3.4, consider $H_0 F = F/\mathrm{rad}(F)$ and the free functor $C_0 := \bigoplus_{a \in J} K(a, -)^{\dim H_0 F(a)}$. Since $H_0 C_0$ and $H_0 F$ are isomorphic, we can use the projectiveness of C_0 to lift the natural transformation $C_0 \to H_0 F$ to a natural transformation $C_0 \to F$ which makes the following square commute, where the vertical arrows represent the quotient natural transformations:

$$\begin{array}{ccc} C_0 & \longrightarrow & F \\ \downarrow & & \downarrow \\ H_0 C_0 & \simeq & H_0 F \end{array}$$

The resulting natural transformation $C_0 \to F$ is an epimorphism (see discussion in 3.2) and hence it is an $\mathcal{S}$-cover. It is also minimal by the same reasoning as in 3.4. It follows that there are enough $\mathcal{S}$-projectives and every functor $F: J \to \mathrm{vect}_K$ has a minimal $\mathcal{S}$-cover, and hence also a minimal $\mathcal{S}$-projective resolution. We refer to

$\mathcal{S}$-covers and $\mathcal{S}$-projective resolutions simply as **covers** and **resolutions**. Since every functor admits a minimal resolution, $\mathcal{S}$-Betti diagrams in all degrees are always defined for every functor. We also refer to them simply as **Betti diagrams** and denote them as $\beta^d F\colon J \to \mathbb{N}$, omitting the symbol $\mathcal{S}$. A consequence of the above construction of a minimal cover of F is the equality:

$$\beta^0 F = \dim H_0 F.$$

The sum $\sum_{a\in J} \beta^0 F(a)$ is called the **number of generators of** F.

The Betti diagrams of F are not independent from each other. For example, consider a differential in a minimal resolution of F:

$$\overbrace{\bigoplus_{b\in \operatorname{supp}(\beta^{d+1}F)} K(b,-)^{\beta^{d+1}F(b)}}^{C_{d+1}} \xrightarrow{\partial} \overbrace{\bigoplus_{a\in \operatorname{supp}(\beta^{d}F)} K(a,-)^{\beta^{d}F(a)}}^{C_{d}}$$

By minimality, each summand $K(b,-)$ of C_{d+1} must map nontrivially onto at least one summand $K(a,-)$ of C_d. This means that every element in $\operatorname{supp}(\beta^{d+1}F)$ is bounded below by some element in $\operatorname{supp}(\beta^d F)$. Relations between Betti diagrams are easier to describe when the indexing poset J is an **upper semilattice**. In this case, to calculate Betti diagrams one can use Koszul complexes. Describing this is the content of the rest of this section.

3.6 Upper semilattices Recall that the **join** of a subset $S \subseteq J$ is an element in J, denoted by $\bigvee_J S$ or $\bigvee S$ when the poset is understood, satisfying the following universal property: $s \preccurlyeq \bigvee S$ for every s in S, and, if $s \preccurlyeq a$ in J for every s in S, then $\bigvee S \preccurlyeq a$. The join is the categorical coproduct in J. Dually, the **meet** of S is an element in J, denoted by $\bigwedge_J S$ or $\bigwedge S$, satisfying the following universal property: $\bigwedge S \preccurlyeq s$ for every s in S, and, if $a \preccurlyeq s$ in J for every s in S, then $a \preccurlyeq \bigwedge S$. When they exist, joins and meets are unique. If every nonempty subset of J has a join, then J is called an **upper semilattice**. If J is an upper semilattice, then every subset S which is bounded below (i.e., for which there exists a in J such that $a \preccurlyeq s$ for every s in S) also has a meet, given by $\bigvee\{a \in J \mid S \text{ is bounded below by } a\}$.

A subset $L \subseteq J$ of an upper semilattice is called a **sublattice** if, for every nonempty subset $S \subseteq L$, the join $\bigvee S$ belongs to L. For $S \subseteq J$, we denote by $\langle S\rangle := \left\{\bigvee T \mid T \neq \varnothing, T \subseteq S\right\} \subseteq J$ the smallest sublattice containing S.

3.7 Koszul complexes Let a be an element in J. Recall that an element s in J is called a **parent** of a if $s \prec a$ and there is no element b in J such that $s \prec b \prec a$. We also say that a **covers** s. We denote the set of parents of a by $\mathcal{U}_J(a)$ or $\mathcal{U}(a)$ when the ambient poset is understood.

Suppose that a has the following property: every subset $S \subseteq \mathcal{U}(a)$ that is bounded below admits a meet $\bigwedge S$. If J is an upper semilattice, then every element $a \in J$ satisfies this property. Fix a total order $<$ on the set of parents $\mathcal{U}(a)$. We then assign to every functor $F\colon J \to \mathrm{vect}_K$ a chain complex of K-vector spaces, called the **Koszul**

complex of F at a, defined for all $d \geq 0$ by

$$(\mathcal{K}_a F)_d := \begin{cases} F(a) & \text{if } d = 0, \\ \displaystyle\bigoplus_{\substack{S \subseteq \mathcal{U}(a),\ |S|=d \\ S \text{ has lower bound}}} F(\bigwedge S) & \text{if } d > 0. \end{cases}$$

For example, $(\mathcal{K}_a F)_1 = \bigoplus_{s \in \mathcal{U}(a)} F(s)$. We define the differentials as follows:

- For $d = 0$, define $\partial\colon (\mathcal{K}_a F)_1 \to (\mathcal{K}_a F)_0 = F(a)$ as the linear map which on the summand $F(s)$ in $\bigoplus_{s \in \mathcal{U}(a)} F(s) = (\mathcal{K}_a F)_1$ is given by $F(s \prec a)$.
- For $d > 0$, define $\partial\colon (\mathcal{K}_a F)_{d+1} \to (\mathcal{K}_a F)_d$ as the alternating sum $\partial = \sum_{i=0}^{d} (-1)^i \partial_i$, where $\partial_i\colon (\mathcal{K}_a F)_{d+1} \to (\mathcal{K}_a F)_d$ is the function mapping the summand $F(\bigwedge S)$ in $(\mathcal{K}_a F)_{d+1}$, indexed by $S = \{s_0 < \cdots < s_d\} \subseteq \mathcal{U}(a)$, to the summand $F(\bigwedge(S \backslash \{s_i\}))$ in $(\mathcal{K}_a F)_d$, indexed by $S \backslash \{s_i\} \subseteq \mathcal{U}(a)$, via the function $F(\bigwedge S \preccurlyeq \bigwedge(S \backslash \{s_i\}))$.

The linear functions ∂ form a chain complex as it is standard to verify that the composition of two consecutive such functions is null.

For a natural transformation $f\colon F \to G$, define:

$$(\mathcal{K}_a F)_d \xrightarrow{(\mathcal{K}_a f)_d} (\mathcal{K}_a G)_d \ := \begin{cases} f_a & \text{if } d = 0, \\ \displaystyle\bigoplus_{\substack{S \subseteq \mathcal{U}(a),\ |S|=d \\ S \text{ has lower bound}}} f_{\bigwedge S} & \text{if } d > 0. \end{cases}$$

These linear maps form a chain map denoted by $\mathcal{K}_a f\colon \mathcal{K}_a F \to \mathcal{K}_a G$. The assignment $f \mapsto \mathcal{K}_a f$ is a functor from $\mathrm{Fun}(I, \mathrm{vect}_K)$ to the category of chain complexes called the **Koszul complex at** a. Its fundamental property is the following.

3.8 Theorem *Let J be a finite poset. Let a be an element in J such that every nonempty subset $S \subseteq \mathcal{U}(a)$ that is bounded below admits a meet $\bigwedge S$. Then, for every functor $F\colon J \to \mathrm{vect}_K$ and every $d \geq 0$, the following equality holds:*

$$\beta^d F(a) = \dim H_d(\mathcal{K}_a F).$$

The assumption on the element a in Theorem 3.8 is local, depending only on the parents of a. Under this assumption, which is satisfied for example if J is an upper semilattice (see 3.6), in order to calculate the Betti diagram $\beta^d F(a)$ of a functor F at a, we do not need to construct the minimal resolution of F. Instead, it suffices to calculate the homology of the Koszul complex $\mathcal{K}_a F$, which can be done for each a and each degree d independently.

Proof of Theorem 3.8 The proof is based on three observations and follows closely what is presented in [9, Section 10].

Observation 1. For every natural transformation $f\colon F \to G$, the linear map $H_0(\mathcal{K}_a f)$ is isomorphic to $(H_0 f)_a$. In particular, the vector spaces $H_0(\mathcal{K}_a F)$ and

$(H_0F)(a) = (F/\mathrm{rad}(F))(a)$ are isomorphic. Moreover, if $C_0 \to F$ is a minimal cover, then $H_0(\mathcal{K}_a(C_0 \to F))$ is an isomorphism.

According to Observation 1, $\beta^0 F(a) = \dim(H_0F)(a)$ (see 3.5) coincides with $\dim H_0(\mathcal{K}_a F)$, which is the statement of the theorem in the case $d = 0$. To prove this statement for $d > 0$, we need two additional observations:

Observation 2. For b in J:

$$\dim H_d(\mathcal{K}_a K(b, -)) = \begin{cases} 1 & \text{if } d = 0 \text{ and } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, if $b \not\preccurlyeq a$, then $\mathcal{K}_a K(b, -) = 0$ and the statement holds. Otherwise, if $b = a$, then $(\mathcal{K}_a K(a, -))_0 = K$ and $(\mathcal{K}_a K(a, -))_d = 0$ for $d > 0$, and again the statement holds.

Otherwise, $b \prec a$. Then $(\mathcal{K}_a K(b, -))_0 = K$ and, for every nonempty subset $S \subseteq \mathcal{U}(a)$ that is bounded below, $K(b, \bigwedge S)$ is 1-dimensional if S is a subset of $(b \preccurlyeq \mathcal{U}(a)) := \{s \in \mathcal{U}(a) \mid b \preccurlyeq s\}$, and 0 otherwise. Consequently the complex $\mathcal{K}_a K(b, -)$ is isomorphic to

$$\cdots \to \bigoplus_{\substack{S \subseteq (b \preccurlyeq \mathcal{U}(a)) \\ |S|=2}} K \to \bigoplus_{\substack{S \subseteq (b \preccurlyeq \mathcal{U}(a)) \\ |S|=1}} K \to K,$$

which is the augmented chain complex of the standard $(|b \preccurlyeq \mathcal{U}(a)| - 1)$-dimensional simplex, whose homology is trivial in all degrees. Thus in this case the statement also holds.

Observation 3. Since taking direct sums is an exact operation, the Koszul complex at a is an exact functor in the following sense: if $f : F \to G$ and $g : G \to H$ form an exact sequence, then the following is an exact sequence of chain complexes of vector spaces:

$$\mathcal{K}_a F \xrightarrow{\mathcal{K}_a f} \mathcal{K}_a G \xrightarrow{\mathcal{K}_a g} \mathcal{K}_a H.$$

According to Observation 3, the Koszul complex functor at a commutes with direct sums, which, combined with Observation 2, implies that for a free functor $C_0 = \bigoplus_{b \in J} K(b, -)^{\beta(b)}$,

$$\dim H_d(\mathcal{K}_a C_0) = \begin{cases} \beta(a) & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $F : J \to \mathrm{vect}_K$ be a functor and $C_0 \to F$ its minimal cover. This minimal cover fits into the following exact sequence:

$$0 \longrightarrow Z \longrightarrow C_0 \longrightarrow F \longrightarrow 0,$$

which, according to Observation 3, leads to an exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{K}_a Z \longrightarrow \mathcal{K}_a C_0 \longrightarrow \mathcal{K}_a F \longrightarrow 0,$$

which in turn leads to a long exact sequence of vector spaces:

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (X.center) \tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
H_1(\mathcal{K}_a Z) \arrow[r] & H_1(\mathcal{K}_a C_0) \arrow[r] & H_1(\mathcal{K}_a F) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Y.center) \tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] & \\
H_0(\mathcal{K}_a Z) \arrow[r] & H_0(\mathcal{K}_a C_0) \arrow[r] & H_0(\mathcal{K}_a F) \arrow[r] & 0.
\end{tikzcd}$$

According to Observation 1, the function $H_0(\mathcal{K}_a C_0) \to H_0(\mathcal{K}_a F)$ is an isomorphism, and $H_0(\mathcal{K}_a Z)$ is isomorphic to $H_0(Z)$, whose dimension is $\beta^1 F(a)$. Observation 2 gives the vanishing of $H_d(\mathcal{K}_a C_0)$ for $d \geq 1$. Consequently, for $d \geq 1$, the map $H_d(\mathcal{K}_a F) \to H_{d-1}(\mathcal{K}_a Z)$ is an isomorphism. The case $d = 1$ then gives the equality between $\dim H_0(\mathcal{K}_a Z) = \beta^0 Z(a) = \beta^1 F(a)$ and $\dim H_1(\mathcal{K}_a F)$, which is the statement of the theorem for $d = 1$. The theorem for $d > 1$ follows by induction by applying what we already have proven to the functor Z. $\square$

Theorem 3.8, combined with the long exact sequence argument in its proof, implies the following.

3.9 Corollary *Let J be a finite upper semilattice and $0 \to F_0 \to F_1 \to F_2 \to 0$ an exact sequence in $\mathrm{Fun}(J, \mathrm{vect}_K)$. Let a be an element in J such that $\beta^d F_0(a) = 0$ for every $d \geq 0$. Then $\beta^d F_1(a) = \beta^d F_2(a)$ for every $d \geq 0$.*

4 Filtrations, Subfunctors, and Their Standard Betti Diagrams

In this section, we discuss in more depth resolutions and Betti diagrams of certain filtrations. Recall that a functor $F\colon J \to \mathrm{vect}_K$ is called a **filtration** if, for all $a \preccurlyeq b$ in J, $F(a \preccurlyeq b)$ is a monomorphism. For example, free functors and constant functors are filtrations, and in particular the constant functor $K_J\colon J \to \mathrm{vect}_K$, whose values are the 1-dimensional vector space K, is a filtration. Since subfunctors of a filtration are filtrations, the kernel of a cover $C_0 \to F$ is a filtration. Two results of this section are of particular interest to us. One is Corollary 4.3, which describes relations between the supports of Betti diagrams of a filtration whose 0^{th} Betti diagram has support bounded below. The other one is Corollary 4.9, which describes a similar, but weaker, relation for any subfunctor of the constant functor K_J, even if its 0^{th} Betti diagram does not have a support bounded below.

The following theorem follows from results in [9] (see also [25, Lemma 2.1]).

4.1 Theorem *Let J be a finite upper semilattice and $F\colon J \to \mathrm{vect}_K$ a functor. Then the following containment holds:*

$$\bigcup_{d\geq 0} \operatorname{supp}(\beta^d F) \subseteq \langle \operatorname{supp}(\beta^0 F) \cup \operatorname{supp}(\beta^1 F)\rangle.$$

Proof Consider a minimal presentation of F given by an exact sequence:

$$\bigoplus_{a\in\operatorname{supp}(\beta^1 F)} K(a,-)^{\beta^1 F(a)} \xrightarrow{\ \partial\ } \bigoplus_{a\in\operatorname{supp}(\beta^0 F)} K(a,-)^{\beta^0 F(a)} \longrightarrow F \longrightarrow 0,$$

Using the vocabulary of [9], we can discretize the functors of this sequence by the sublattice $\langle \operatorname{supp}(\beta^0 F) \cup \operatorname{supp}(\beta^1 F)\rangle$. By [9, Corollary 10.18.(2)], we conclude that the Betti diagrams of F indexed by J are isomorphic to those of F restricted to the sublattice $\langle \operatorname{supp}(\beta^0 F) \cup \operatorname{supp}(\beta^1 F)\rangle \subset J$, and consequently

$$\bigcup_{d\geq 0} \operatorname{supp}(\beta^d F) \subseteq \langle \operatorname{supp}(\beta^0 F) \cup \operatorname{supp}(\beta^1 F)\rangle.$$

□

As a consequence, if $\beta^d F(a) \neq 0$, then there is a subset $S \subseteq \operatorname{supp}(\beta^0 F) \cup \operatorname{supp}(\beta^1 F)$ for which $a = \bigvee S$. This gives a considerable restriction on what elements of the upper semilattice J can belong to $\operatorname{supp}(\beta^d F)$ for $d > 1$.

4.2 Corollary *Let J be a finite upper semilattice and $F\colon J \to \mathrm{vect}_K$ a functor for which $\operatorname{supp}(\beta^0 F)$ is bounded below (the meet $\bigwedge \operatorname{supp}(\beta^0 F)$ exists). Then*

$$\cdots \subseteq \langle \operatorname{supp}(\beta^3 F)\rangle \subseteq \langle \operatorname{supp}(\beta^2 F)\rangle \subseteq \langle \operatorname{supp}(\beta^1 F)\rangle.$$

Proof Let $b_0 = \bigwedge \operatorname{supp}(\beta^0 F)$. For all a in $\operatorname{supp}(\beta^0 F)$, we have $a \succcurlyeq b_0$, so there is a monomorphism $K(a,-) \subseteq K(b_0,-)$. These monomorphisms, for all a in $\operatorname{supp}(\beta^0 F)$, fit into the following pushout square where the top horizontal arrow represents a minimal cover of F:

$$\begin{tikzcd}
\bigoplus_{a\in\operatorname{supp}(\beta^0 F)} K(a,-)^{\beta^0 F(a)} \arrow[r] \arrow[d, hook] & F \arrow[d, hook] \\
\bigoplus_{a\in\operatorname{supp}(\beta^0 F)} K(b_0,-)^{\beta^0 F(a)} \arrow[r] & G
\end{tikzcd}$$

Then the natural transformation represented by the bottom horizontal arrow is a minimal cover of G. Furthermore, since the kernels of the top and bottom horizontal natural transformations coincide, we get:

$$\beta^d G(a) = \begin{cases} \sum_{b\in J} \beta^0 F(b) & \text{if } d = 0 \text{ and } a = b_0, \\ 0 & \text{if } d = 0 \text{ and } a \neq b_0, \\ \beta^d F(a) & \text{if } d > 0. \end{cases}$$

In particular $\mathrm{supp}(\beta^0 G) = \{b_0\}$ and $\mathrm{supp}(\beta^d G) = \mathrm{supp}(\beta^d F)$ for every $d \geq 1$. Since every element in $\mathrm{supp}(\beta^2 G)$ is bounded by some element in $\mathrm{supp}(\beta^1 G)$, and every element in $\mathrm{supp}(\beta^1 G)$ is bounded by some element in $\mathrm{supp}(\beta^0 G) = \{b_0\}$, we can use Theorem 4.1 to conclude that $\mathrm{supp}(\beta^2 G) \subseteq \langle \mathrm{supp}(\beta^1 G)\rangle$ and consequently $\langle \mathrm{supp}(\beta^2 G)\rangle \subseteq \langle \mathrm{supp}(\beta^1 G)\rangle$. We then proceed by induction on d. □

For a filtration, Corollary 4.2 can be extended to include also the 0^{th} Betti diagram.

4.3 Corollary *Let J be a finite upper semilattice and $F\colon J \to \mathrm{vect}_K$ a filtration such that $\mathrm{supp}(\beta^0 F)$ is bounded below. Then*

$$\cdots \subseteq \langle \mathrm{supp}(\beta^2 F)\rangle \subseteq \langle \mathrm{supp}(\beta^1 F)\rangle \subseteq \langle \mathrm{supp}(\beta^0 F)\rangle.$$

Proof The assumptions on F are equivalent to F being a subfunctor of $G := K(b, -)^n$ for some b in J and a natural number n. Consider the exact sequence $F \hookrightarrow G \to G/F$. Since G is free, $\beta^1(G/F)(a) \leq \beta^0 F(a)$ and $\beta^{d+1}(G/F)(a) = \beta^d F(a)$ for $d > 0$ and every a in J. Consequently, $\mathrm{supp}(\beta^1(G/F)) \subseteq \mathrm{supp}(\beta^0 F)$ and $\mathrm{supp}(\beta^{d+1}(G/F)) = \mathrm{supp}(\beta^d F)$ for $d > 0$. By applying Corollary 4.2 to the quotient G/F, we obtain the desired containments. For example,

$$\langle \mathrm{supp}(\beta^1 F)\rangle = \langle \mathrm{supp}(\beta^2 (G/F))\rangle \subseteq \langle \mathrm{supp}(\beta^1 (G/F))\rangle \subseteq \langle \mathrm{supp}(\beta^0 F)\rangle.$$

□

4.4 Subfunctors Next we focus on subfunctors of the constant functor $K_J\colon J \to \mathrm{vect}_K$. Given a subfunctor F of K_J, we denote by $F \subseteq K_J$ the inclusion natural transformation. Note that a functor $F\colon J \to \mathrm{vect}_K$ is isomorphic to a subfunctor of K_J if, and only if, F is a filtration and $\dim F(a) \leq 1$ for every a in J (compare with assumptions in Corollary 4.3). We denote by $\mathrm{Sub}(J)$ the set of all subfunctors of K_J. We consider two ways of parameterizing this set by posets: via upsets and antichains.

4.5 Upsets An **upset** of a poset $(J, \preccurlyeq)$ is a subset $U \subseteq J$ such that, for all a in U and $b \succcurlyeq a$, the element b is in U. The set of upsets of J is denoted $\mathrm{Up}(J)$ and comes naturally equipped with a distributive lattice structure where the order relation is the inclusion $\subseteq$, joins are unions, and meets are intersections. An example of an upset is the **support** of a subfunctor $F \subseteq K_J$, defined as the subset $\mathrm{supp}(F) := \{a \in J \mid F(a) \neq 0\}$. In fact, the function $F \mapsto \mathrm{supp}(F)$ is a bijection between the collections of subfunctors of K_J and upsets in J. Its inverse sends an upset U to the subfunctor $K_U \subseteq K_J$ defined as, for all a in J,

$$K_U(a) = \begin{cases} K & \text{if } a \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $K_U \subseteq K_J$ is the unique subfunctor for which $\mathrm{supp}(K_U) = U$. The inclusion of subfunctors $F \subseteq G \subseteq K_J$ defines a poset relation that makes this bijection a poset isomorphism, between $(\mathrm{Sub}(J), \subseteq)$ and $(\mathrm{Up}(J), \subseteq)$. We are however more interested in the opposite poset to $(\mathrm{Sub}(J), \subseteq)$. This is because of its relation to antichains.

4.6 Antichains An **antichain** of a poset $(J, \preccurlyeq)$ is a subset of J whose elements are pairwise incomparable. In particular, singletons of J are antichains. The set of antichains of J is denoted by $\mathrm{Anti}(J)$. Given an upset U of J, the set $\mathrm{Min}(U)$ of its minimal elements is an antichain. In fact, the function $U \mapsto \mathrm{Min}(U)$ from upsets to antichains is a bijection, and so we could equip antichains with the induced poset structure from $\mathrm{Up}(J)$. However, we would like for the subposet of singletons in $\mathrm{Anti}(J)$ to coincide with the poset J. For this reason, we define the relation on antichains by the opposite order on upsets. Thus, given two upsets U and V, we write $\mathrm{Min}(U) \preccurlyeq \mathrm{Min}(V)$ if $V \subseteq U$. Explicitly, for two antichains S and T in J, the relation $S \preccurlyeq T$ holds if, and only if, for every t in T, there is s in S such that $s \preccurlyeq t$ in J. In this way, $(\mathrm{Anti}(J), \preccurlyeq)$ is a distributive lattice whose restriction to singletons coincides with J.

To a subfunctor $F \subseteq K_J$ we associate the antichain $\mathrm{Min}\,\mathrm{supp}(F)$, which coincides with $\mathrm{supp}(\beta^0 F)$. The function $F \mapsto \mathrm{supp}(\beta^0 F)$ is a bijection between the collection $\mathrm{Sub}(J)$ of subfunctors of K_J and the set of antichains $\mathrm{Anti}(J)$. Its inverse sends an antichain S to the subfunctor $K(S, -) := K_{(S \preccurlyeq J)} \subseteq K_J$ where $(S \preccurlyeq J)$ denotes the upset $\{a \in J \mid \exists s \in S, s \preccurlyeq a\}$. This induces a distributive lattice structure on subfunctors of K_J: given two subfunctors $F, G \subseteq K_J$, we write $F \preccurlyeq G$ if $\mathrm{supp}(\beta^0 F) \preccurlyeq \mathrm{supp}(\beta^0 G)$, which is equivalent to the inclusion $\mathrm{supp}(F) \supseteq \mathrm{supp}(G)$. We denote this poset by $(\mathrm{Sub}(J), \preccurlyeq)$.

4.7 Global Koszul complexes We now discuss free resolutions of subfunctors of K_J under the assumption that J is an upper semilattice. Let $F \subseteq K_J$ be a subfunctor. Fix a total order $<$ on the antichain $\mathrm{supp}(\beta^0 F)$. This order is used to construct a chain complex called **global Koszul complex** of F. For $d \geq 0$, define:

$$(\mathcal{K}F)_d := \bigoplus_{\substack{S \subseteq \mathrm{supp}(\beta^0 F)\\ |S| = d+1}} K\left(\bigvee S, -\right).$$

For example, $(\mathcal{K}F)_0 = \bigoplus_{s \in \mathrm{supp}(\beta^0 F)} K(s, -)$, which coincides with the minimal cover of F (see 3.5). Set $\partial \colon (\mathcal{K}F)_{d+1} \to (\mathcal{K}F)_d$ to be the alternating sum $\partial = \sum_{i=0}^{d+1} (-1)^i \partial_i$, where $\partial_i \colon (\mathcal{K}F)_{d+1} \to (\mathcal{K}F)_d$ is the function mapping the summand $K(\bigvee S, -)$ in $(\mathcal{K}F)_{d+1}$, indexed by $S = \{s_0 < \cdots < s_{d+1}\} \subseteq \mathrm{supp}(\beta^0 F)$, to the summand $K(\bigvee(S \setminus \{s_i\}), -)$ in $(\mathcal{K}F)_d$, indexed by $S \setminus \{s_i\} \subseteq \mathrm{supp}(\beta^0 F)$, via the inclusion $K(\bigvee S, -) \subseteq K(\bigvee(S \setminus \{s_i\}), -)$. Finally define $\partial \colon (\mathcal{K}F)_0 \to F$ to be a minimal cover of F. It is standard to verify that the natural transformations ∂ form a chain complex.

4.8 Proposition *Let J be a finite upper semilattice and $F \subseteq K_J$ be a subfunctor. Then $\mathcal{K}F \to F$ is a free resolution of F.*

Proof We need to show that, for every a in J, the complex of vector spaces $(\mathcal{K}F)(a) \to F(a)$ is exact. Consider the set $T := \{s \in \mathrm{supp}(\beta^0 F) \mid s \preccurlyeq a\}$. If this set is empty, then the complex $(\mathcal{K}F)(a) \to F(a)$ is trivial and hence it is exact. Otherwise, the complex $(\mathcal{K}F)(a) \to F(a)$ is isomorphic to

$$\cdots \to \bigoplus_{\substack{S\subseteq T\\ |S|=2}} K \to \bigoplus_{\substack{S\subseteq T\\ |S|=1}} K \to K,$$

which is the augmented chain complex of the standard $(|T|-1)$-dimensional simplex, whose homology is trivial in all degrees. □

The global Koszul complex $\mathcal{K}F \to F$ is a free resolution of F which may fail however to be minimal. A minimal resolution of F is a direct summand of the global Koszul complex, which gives:

4.9 Corollary *Let J be a finite upper semilattice and $F \subseteq K_J$ be a subfunctor. Then the following containment holds:*

$$\bigcup_{d\geq 0} \operatorname{supp}(\beta^d F) \subseteq \langle \operatorname{supp}(\beta^0 F)\rangle.$$

5 Constructing Relative Homological Algebra

Let $(I, \leq)$ be a finite poset. In this section we present a strategy for defining an independent (see 2.1) and acyclic (see 2.3) collection for relative homological algebra in the category $\operatorname{Fun}(I, \operatorname{vect}_K)$. Our starting point and the standard assumption throughout the entire section is a functor $\mathcal{T}\colon J^{\mathrm{op}} \to \operatorname{Fun}(I, \operatorname{vect}_K)$ where $(J, \preccurlyeq)$ is a finite poset.

We define the collection $\mathcal{P} := \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$. Although no structure on the collection $\mathcal{P}$ was needed in our description of the homological algebra relative to $\mathcal{P}$ in Sect. 2, in this section we illustrate the advantage of having the grading on $\mathcal{P}$ given by the poset structure on J. The functor $\mathcal{T}$ is used to translate between the $\mathcal{P}$-relative homological algebra on $\operatorname{Fun}(I, \operatorname{vect}_K)$ and the standard homological algebra on $\operatorname{Fun}(J, \operatorname{vect}_K)$. This translation is done via a pair of adjoint functors.

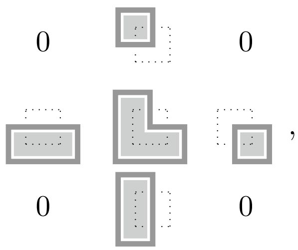
5.1 Example Throughout this section, we consider the following running example. Let $(I, \leq)$ be the set $\{0, 1\}^2$ with the product order, J the set $\{(u, v) \in I^2 \mid u \leq v\}$ with the product order, and

$$\mathcal{T}\colon \begin{cases} J^{\mathrm{op}} \to \operatorname{Fun}(I, \operatorname{vect}_K) \\ (u, v) \mapsto \operatorname{coker}(K(v, -) \to K(u, -)) \end{cases}$$

the translation or **parameterization** functor. Explicitly, we have the Hasse diagrams (where the order goes up and to the right)

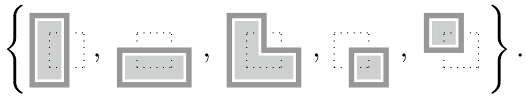
$$
I = \begin{tikzcd} 01 \arrow[r, dash, dotted] & 11 \\ 00 \arrow[r, dash, dotted] \arrow[u, dash, dotted] & 10 \arrow[u, dash, dotted] \end{tikzcd} \quad \text{and} \quad J = \begin{tikzcd} 01,01 \arrow[r, dash, dotted] & 01,11 \arrow[r, dash, dotted] & 11,11 \\ 00,01 \arrow[r, dash, dotted] \arrow[u, dash, dotted] & 00,11 \arrow[r, dash, dotted] \arrow[u, dash, dotted] & 10,11 \arrow[u, dash, dotted] \\ 00,00 \arrow[r, dash, dotted] \arrow[u, dash, dotted] & 00,10 \arrow[r, dash, dotted] \arrow[u, dash, dotted] & 10,10 \arrow[u, dash, dotted] \end{tikzcd}.
$$

The image of J by $\mathcal{T}$ is then



where we have represented each nonzero functor $\mathcal{T}(a) \colon I \to \mathrm{vect}_K$ in $\mathcal{P}$ by their support: the dotted squares correspond to the poset I, the shaded vertices correspond to the support of the functor, where the functor is equal to K, and the transition functions are all implicitly of maximal rank. The position of each functor $\mathcal{T}(a)$ corresponds to the position of a in the Hasse diagram of J.

Finally, we have the collection

$\mathcal{P} =$ 

This example is a specific instance of **lower hooks**, introduced in [6], which we study in general in Example F.

5.2 Adjunction The functor $\mathcal{T}$ induces the following pair of adjoint functors:

$$
\mathrm{Fun}(I, \mathrm{vect}_K) \underset{\mathcal{L}}{\overset{\mathcal{R}}{\rightleftarrows}} \mathrm{Fun}(J, \mathrm{vect}_K)
$$

where

- $\mathcal{R} := \mathrm{Nat}_I(\mathcal{T}, -)$ assigns to $M \colon I \to \mathrm{vect}_K$ the functor $\mathrm{Nat}_I(\mathcal{T}(-), M) : J \to \mathrm{vect}_K$,
- $\mathcal{L}$ assigns to $F \colon J \to \mathrm{vect}_K$ the following colimit in $\mathrm{Fun}(I, \mathrm{vect}_K)$:

$$
\mathcal{L}F := \mathrm{colim}\left(\bigoplus_{a_0 \prec a_1 \text{ in } J} \mathcal{T}(a_1) \otimes F(a_0) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \bigoplus_{a \text{ in } J} \mathcal{T}(a) \otimes F(a) \right),
$$

where the summand $\mathcal{T}(a_1) \otimes F(a_0)$, indexed by $a_0 \prec a_1$, is mapped

- by d_0 to the summand $\mathcal{T}(a_1) \otimes F(a_1)$ via the morphism $\mathrm{id}_{\mathcal{T}(a_1)} \otimes F(a_0 \prec a_1)$;
- by d_1 to the summand $\mathcal{T}(a_0) \otimes F(a_0)$ via the morphism $\mathcal{T}(a_0 \prec a_1) \otimes \mathrm{id}_{F(a_0)}$.

5.3 Proposition *The functor $\mathcal{L}$ is left adjoint to $\mathcal{R}$.*

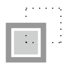
Proof Let $F\colon J \to \mathrm{vect}_K$ and $M\colon I \to \mathrm{vect}_K$ be two functors. By the universal property of the colimit and the tensor-hom adjunction, the set $\mathrm{Nat}_I(\mathcal{L}F, M)$ is in natural bijection with the set of sequences of linear maps $\{f_a\colon F(a) \to \mathrm{Nat}_I(\mathcal{T}(a), M)\}_{a\in J}$ making the following square commute for every $a_0 \prec a_1$ in J:

$$\begin{tikzcd}
F(a_0) \arrow[r, "f_{a_0}"] \arrow[d, "F(a_0\prec a_1)"'] & \mathrm{Nat}_I(\mathcal{T}(a_0), M) \arrow[d, "\mathrm{Nat}_I(\mathcal{T}(a_0\prec a_1), M)"] \\
F(a_1) \arrow[r, "f_{a_1}"] & \mathrm{Nat}_I(\mathcal{T}(a_1), M)
\end{tikzcd}$$

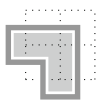
Thus this set of sequences is in bijection with $\mathrm{Nat}_J(F, \mathrm{Nat}_I(\mathcal{T}, M)) = \mathrm{Nat}_J(F, \mathcal{R}M)$. In this way we get a natural isomorphism between $\mathrm{Nat}_I(\mathcal{L}F, M)$ and $\mathrm{Nat}_J(F, \mathcal{R}M)$, which gives the desired adjunction. □

Let a be in J and consider the free functor $K(a, -)\colon J \to \mathrm{vect}_K$. By the adjunction between $\mathcal{L}$ and $\mathcal{R}$, for every $M\colon I \to \mathrm{vect}_K$, the set $\mathrm{Nat}_I(\mathcal{L}K(a, -), M)$ is naturally isomorphic to the set $\mathrm{Nat}_J(K(a, -), \mathcal{R}M)$, which is in natural bijection with $\mathcal{R}M(a) = \mathrm{Nat}_I(\mathcal{T}(a), M)$. Consequently $\mathcal{L}K(a, -)$ and $\mathcal{T}(a)$ are isomorphic functors for every a in J, and the collection $\mathcal{P} = \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$ can be identified with $\{\mathcal{L}K(a, -) \mid a \in J,\ \mathcal{L}K(a, -) \neq 0\}$.

5.4 Example Continuing our running Example 5.1, consider the functor $M\colon I \to \mathrm{vect}_K$ represented as follows:



Then, using the same representation conventions for functors indexed by J, the functor $\mathcal{R}M$ is represented as



5.5 Adjunction unit and counit For $M\colon I \to \mathrm{vect}_K$, the **counit** $\varepsilon_M\colon \mathcal{L}\mathcal{R}M \to M$ is the natural transformation adjoint to $\mathrm{id}\colon \mathcal{R}M \to \mathcal{R}M$. For $F\colon J \to \mathrm{vect}_K$, the **unit** $\eta_F\colon F \to \mathcal{R}\mathcal{L}F$ is the natural transformation adjoint to $\mathrm{id}\colon \mathcal{L}F \to \mathcal{L}F$. If $F = K(a, -)$ for some a in J, then η_F is also denoted by η_a. The adjunction between $\mathcal{L}$ and $\mathcal{R}$ implies the commutativity of the following diagrams, which in particular implies that $\mathcal{R}\varepsilon_M$ and $\varepsilon_{\mathcal{L}F}$ are epimorphisms, and $\eta_{\mathcal{R}M}$ and $\mathcal{L}\eta_F$ are monomorphisms:

$$\begin{tikzcd} \mathcal{R}M \arrow[r, "\eta_{\mathcal{R}M}"] \arrow[dr, bend right, "\mathrm{id}"'] & \mathcal{R}\mathcal{L}\mathcal{R}M \arrow[d, "\mathcal{R}\varepsilon_M"] \\ & \mathcal{R}M \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}\mathcal{R}\mathcal{L}F \arrow[d, "\varepsilon_{\mathcal{L}F}"'] & \mathcal{L}F \arrow[l, "\mathcal{L}\eta_F"'] \arrow[dl, bend left, "\mathrm{id}"] \\ \mathcal{L}F & \end{tikzcd}$$

The functor $\mathcal{R}$ can be used to translate between $\mathcal{P}$-exactness in $\mathrm{Fun}(I, \mathrm{vect}_K)$ and the standard exactness in $\mathrm{Fun}(J, \mathrm{vect}_K)$:

5.6 Proposition *Let I and J be finite posets. A pair of natural transformations $f : M \to N$ and $g : N \to L$ in $\mathrm{Fun}(I, \mathrm{vect}_K)$ form a $\mathcal{P}$-exact sequence (see 2.2) if, and only if, the natural transformations $\mathcal{R}f$ and $\mathcal{R}g$ form an exact sequence in $\mathrm{Fun}(J, \mathrm{vect}_K)$.*

Proof By definition of the functor $\mathcal{R} = \mathrm{Nat}_I(\mathcal{T}, -)$, the pair $\mathcal{R}f$ and $\mathcal{R}g$ forms an exact sequence if, and only if, $\mathrm{Nat}_I(\mathcal{T}(a), f)$ and $\mathrm{Nat}_I(\mathcal{T}(a), g)$ form an exact sequence for all a in J, which corresponds to $\mathcal{P}$-exactness in $\mathrm{Fun}(I, \mathrm{vect}_K)$. □

5.7 Corollary *A natural transformation $f : M \to N$ in $\mathrm{Fun}(I, \mathrm{vect}_K)$ is a $\mathcal{P}$-epimorphism if, and only if, $\mathcal{R}f$ is an epimorphism in $\mathrm{Fun}(J, \mathrm{vect}_K)$.*

For example, consider a functor $M : I \to \mathrm{vect}_K$ and the natural transformation $\varepsilon_M : \mathcal{L}\mathcal{R}M \to M$. Since $\mathcal{R}\varepsilon_M$ is an epimorphism, ε_M is therefore a $\mathcal{P}$-epimorphism.

To translate between minimal $\mathcal{P}$-covers in $\mathrm{Fun}(I, \mathrm{vect}_K)$ and standard minimal covers in $\mathrm{Fun}(J, \mathrm{vect}_K)$, an additional assumption on the functor $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is required. Here is the key definition of our paper.

5.8 Thinness The functor $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is called **thin** if, for every a in J, the unit natural transformation $\eta_a : K(a, -) \to \mathcal{R}\mathcal{L}K(a, -)$ is an epimorphism.

Our primary examples of thin functors are given by the following result:

5.9 Proposition *Let I and J be finite posets and $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ a functor. Suppose that*

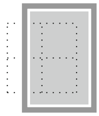
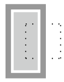
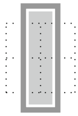
- *for all a in J, the functor $\mathcal{T}(a)$ has at most one generator, i.e. $\sum_{v \in I} \beta^0 \mathcal{T}(a)(v) \leq 1$.*
- *for all a, b in J, $\mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a)) \neq 0$ only if $a \preccurlyeq b$.*

Then the functor $\mathcal{T}$ is thin.

Proof Let a be in J. If $\mathcal{T}(a) = 0$, then $\mathcal{R}\mathcal{L}K(a, -) = \mathrm{Nat}(\mathcal{T}(-), \mathcal{T}(a)) = 0$, so the unit natural transformation $\eta_a : K(a, -) \to \mathcal{R}\mathcal{L}K(a, -)$ is surjective.

Otherwise, let b be another element of J. If $\mathcal{T}(b) = 0$, then $\mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a)) = 0$. If $a \not\preccurlyeq b$, then by hypothesis we also have $\mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a)) = 0$. Otherwise, we have $a \preccurlyeq b$ and $\mathcal{T}(b) \neq 0$. Using the hypothesis on $\mathcal{T}$, write $\mathrm{supp}(\beta^0 \mathcal{T}(b)) = \{x_b\}$. Every natural transformation $\varphi : \mathcal{T}(b) \to \mathcal{T}(a)$ is then determined by $\varphi_{x_b} : \mathcal{T}(b)(x_b) \to \mathcal{T}(a)(x_b)$. Since $\dim \mathcal{T}(b)(x_b) = 1$ and $\dim \mathcal{T}(a)(x_b) \leq 1$, we get $\dim \mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a)) \leq 1$. Moreover, since $a \preccurlyeq b$, we have $\dim K(a, b) = 1$, and so the natural map $\eta_a(b) : K(a, b) \to \mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a))$ is a surjection. Thus the unit natural transformation $\eta_a : K(a, -) \to \mathcal{R}\mathcal{L}K(a, -) = \mathrm{Nat}(\mathcal{T}(-), \mathcal{T}(a))$ is surjective, and we conclude that $\mathcal{T}$ is thin. □

5.10 Example In our running Example 5.1, the functor $\mathcal{T}$ is thin. Lower hooks fit into the setting of Proposition 5.9, but we can also check thinness by hand: for instance, for $a = (00, 10)$ in J, we have

 $= K(a,-) \xrightarrow{\eta_a} \mathcal{R}\mathcal{L}K(a,-) = \mathcal{R}$  $=$  ,

so the unit natural transformation $\eta_a \colon K(a,-) \to \mathcal{R}\mathcal{L}K(a,-)$ is indeed an epimorphism.

Among thin functors there are functors that satisfy a stronger requirement.

5.11 Flatness The functor $\mathcal{T} \colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is called **flat** if the unit natural transformation $\eta_a \colon K(a,-) \to \mathcal{R}\mathcal{L}K(a,-)$ is an isomorphism for every a in J for which $\mathcal{T}(a) \neq 0$.

Every flat functor is thin. Since both left and right adjoints commute with direct sums, if $\mathcal{T}$ is thin, then, for every free functor C_0 in $\mathrm{Fun}(J, \mathrm{vect}_K)$, the unit natural transformation $\eta_{C_0} \colon C_0 \to \mathcal{R}\mathcal{L}C_0$ is also an epimorphism. If $\mathcal{T}$ is flat and C_0 in $\mathrm{Fun}(J, \mathrm{vect}_K)$ is a free functor, then the unit natural transformation $\eta_{C_0} \colon C_0 \to \mathcal{R}\mathcal{L}C_0$ is an isomorphism if, and only if, $\mathcal{T}(a) \neq 0$ for every a in $\mathrm{supp}(\beta^0 C_0)$.

5.12 Proposition *Let I and J be finite posets. If the functor $\mathcal{T} \colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is thin (resp. flat), then, for every subposet $L \subseteq J$, the restriction of $\mathcal{T}$ to $L^{\mathrm{op}} \subseteq J^{\mathrm{op}}$ is also thin (resp. flat).*

Proof For all a in J, the functors $\mathcal{R}\mathcal{L}K(a,-)$, $\mathcal{R}\mathcal{T}(a)$, and $\mathrm{Nat}_I(\mathcal{T}, \mathcal{T}(a))$ are isomorphic. Thus $\mathcal{T}$ is thin if, and only if, both of the following conditions are satisfied:

- for $a \not\preccurlyeq b$ in J, $\mathrm{Nat}_I(\mathcal{T}(b), \mathcal{T}(a)) = 0$,
- for $a \preccurlyeq b$ in J, every natural transformation $\mathcal{T}(b) \to \mathcal{T}(a)$ is of the form $\lambda\mathcal{T}(a \preccurlyeq b)$ for some λ in the field K.

This characterization implies that if $\mathcal{T}$ is thin, then its restriction to L^{op} is also thin.

Similarly, the functor $\mathcal{T}$ is flat if, and only if, both of the following conditions are satisfied:

- for $a \not\preccurlyeq b$ in J, $\mathrm{Nat}_I(\mathcal{T}(b), \mathcal{T}(a)) = 0$,
- for $a \preccurlyeq b$ in J, if $\mathcal{T}(a) \neq 0$, then $\mathrm{Nat}_I(\mathcal{T}(b), \mathcal{T}(a))$ is 1-dimensional and $\mathcal{T}(a \preccurlyeq b)$ is nonzero.

As before, this characterization implies that if $\mathcal{T}$ is flat, then its restriction to L^{op} is also flat. □

Thinness is important because it implies that all the elements in the collection $\mathcal{P}$ are indecomposable (as their endomorphism algebras are 1 dimensional), and consequently this collection is independent (see 2.1). Nevertheless we find the following proof insightful.

5.13 Proposition *Let I and J be finite posets. If the functor $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is thin, then the collection $\mathcal{P} = \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$ is independent (see 2.1).*

Proof Choose a in J. The functor $\mathcal{R}\mathcal{L}K(a,-) = \mathrm{Nat}_I(\mathcal{T}, \mathcal{T}(a)) = \mathcal{R}\mathcal{T}(a)$ is the zero functor if, and only if, $\mathcal{T}(a)$ is the zero functor. Thus, if $\mathcal{T}(a)$ is nonzero, then the surjectivity of $\eta_a\colon K(a,-) \to \mathcal{R}\mathcal{L}K(a,-)$ ($\mathcal{T}$ is assumed to be thin) implies that η_a is a minimal cover in $\mathrm{Fun}(J, \mathrm{vect}_K)$, in which case the standard 0$^{\text{th}}$ Betti diagram $\beta^0\mathcal{R}\mathcal{T}(a) = \beta^0\mathcal{R}\mathcal{L}K(a,-)\colon J \to \mathbb{N}$ has the following values:

$$\beta^0\mathcal{R}\mathcal{T}(a)(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\beta\colon J \to \mathbb{N}$ be a function whose support is finite and such that, if $\beta(a) \neq 0$, then $\mathcal{T}(a) \neq 0$. Consider the functor $\bigoplus_{a\in J} \mathcal{T}(a)^{\beta(a)}$, which is a finite sum because J is finite. Since $\mathcal{R}$ is right adjoint, it commutes with finite direct sums and consequently $\mathcal{R}\left(\bigoplus_{a\in J} \mathcal{T}(a)^{\beta(a)}\right)$ is isomorphic to $\bigoplus_{a\in J} \mathcal{R}\mathcal{T}(a)^{\beta(a)}$. According to the calculation above, β is the 0$^{\text{th}}$ Betti diagram of $\mathcal{R}\left(\bigoplus_{a\in J} \mathcal{T}(a)^{\beta(a)}\right)$. The function β is therefore determined uniquely by the isomorphism type of the functor $\bigoplus_{a\in J} \mathcal{T}(a)^{\beta(a)}$, which gives the independence of $\mathcal{T}$. □

If $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is thin, then we also have an effective way of constructing (minimal) $\mathcal{P}$-covers in $\mathrm{Fun}(I, \mathrm{vect}_K)$ using standard (minimal) covers in $\mathrm{Fun}(J, \mathrm{vect}_K)$.

5.14 Theorem *Let I and J be finite posets and $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ a thin functor* (see 5.8). *Then, for every $M\colon I \to \mathrm{vect}_K$, a natural transformation $C_0 \to \mathcal{R}M$ is a cover in* $\mathrm{Fun}(J, \mathrm{vect}_K)$ *if, and only if, its adjoint $\mathcal{L}C_0 \to M$ is a $\mathcal{P}$-cover in* $\mathrm{Fun}(I, \mathrm{vect}_K)$. *Moreover, if $C_0 \to \mathcal{R}M$ is a minimal cover, then its adjoint $\mathcal{L}C_0 \to M$ is a minimal $\mathcal{P}$-cover.*

Proof Since, for every a in J, the functor $\mathcal{L}K(a,-)$ is isomorphic to $\mathcal{T}(a)$ and $\mathcal{L}$ commutes with direct sums, a functor C_0 in $\mathrm{Fun}(J, \mathrm{vect}_K)$ is free if, and only if, the functor $\mathcal{L}C_0$ in $\mathrm{Fun}(I, \mathrm{vect}_K)$ is $\mathcal{P}$-free. Recall that all projectives in $\mathrm{Fun}(J, \mathrm{vect}_K)$ are free.

Choose a functor $M\colon I \to \mathrm{vect}_K$, and a natural transformation $f\colon C_0 \to \mathcal{R}M$ with C_0 a free functor in $\mathrm{Fun}(J, \mathrm{vect}_K)$. Let $g\colon \mathcal{L}C_0 \to M$ be the adjoint of f. The following commuting diagram describes the relation between f and g:

$$\begin{tikzcd} C_0 \arrow[r, "\eta_{C_0}"] \arrow[dr, bend right, "f"'] & \mathcal{R}\mathcal{L}C_0 \arrow[d, "\mathcal{R}g"] \\ & \mathcal{R}M \end{tikzcd}$$

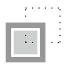
By thinness assumption, the unit $\eta_{C_0}\colon C_0 \to \mathcal{R}\mathcal{L}C_0$ is an epimorphism. Therefore f is an epimorphism if, and only if, $\mathcal{R}g$ is an epimorphism, which, by Proposition 5.6, happens if, and only if, g is a $\mathcal{P}$-epimorphism. Thus f is a cover if, and only if, g is a $\mathcal{P}$-cover.

Next we discuss minimality. Suppose that $f : C_0 \to \mathcal{R}M$ is a minimal cover. Since we already showed that $g : \mathcal{L}C_0 \to M$ is a $\mathcal{P}$-cover, it remains to show its minimality. Let $h : \mathcal{L}C_0 \to \mathcal{L}C_0$ be an endomorphism of g. This endomorphism fits into the following commutative diagram:

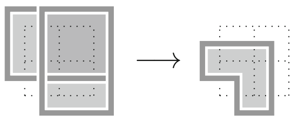
$$\begin{tikzcd}
C_0 \arrow[r, "\eta_{C_0}"] \arrow[dd, dashed] \arrow[rrd, bend left, "f"] & \mathcal{R}\mathcal{L}C_0 \arrow[dd, "\mathcal{R}h"] \arrow[rd, "\mathcal{R}g"] & \\
& & \mathcal{R}M \\
C_0 \arrow[r, "\eta_{C_0}"] \arrow[rru, bend right, "f"'] & \mathcal{R}\mathcal{L}C_0 \arrow[ru, "\mathcal{R}g"'] &
\end{tikzcd}$$

As before, the unit $\eta_{C_0} : C_0 \to \mathcal{R}\mathcal{L}C_0$ is an epimorphism. The dashed arrow exists because C_0 is free. By the minimality of f, this dashed arrow is an isomorphism. The natural transformation $\mathcal{R}h$ is therefore an epimorphism. By Corollary 5.7, $h : \mathcal{L}C_0 \to \mathcal{L}C_0$ is a $\mathcal{P}$-epimorphism. Since $\mathcal{L}C_0$ is $\mathcal{P}$-free, h being a $\mathcal{P}$-epimorphism means that h is an epimorphism. As the values of $\mathcal{L}C_0$ are finite dimensional vector spaces, h is an isomorphism. The natural transformation g is therefore a minimal cover. □

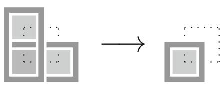
5.15 Example Considering the functor from Example 5.4

$M =$  ,

the minimal cover of $\mathcal{R}M$ is

 ,

that is, $K((00, 10), -) \oplus K((00, 01), -) \to \mathcal{R}M$. Then, by Theorem 5.14, the minimal $\mathcal{P}$-cover of M is obtained by applying $\mathcal{L}$ (see the discussion before 5.4):

 .

Theorem 5.14 has several important consequences, including the acyclicity of the collection $\mathcal{P}$:

5.16 Corollary *Let I and J be finite posets. Suppose the functor $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is thin. Then*

(1) every functor in $\mathrm{Fun}(I, \mathrm{vect}_K)$ admits a minimal $\mathcal{P}$-resolution (see 2.5);
(2) the collection $\mathcal{P} = \{\mathcal{T}(a) \mid a \in J,\ \mathcal{T}(a) \neq 0\}$ is acyclic (see 2.3).

Proof (1) Every functor in $\mathrm{Fun}(J, \mathrm{vect}_K)$ admits a minimal cover (see 3.5). Thus according to Theorem 5.14 every functor in $\mathrm{Fun}(I, \mathrm{vect}_K)$ admits a minimal $\mathcal{P}$-cover, and hence a minimal $\mathcal{P}$-resolution.

(2) Let M be a $\mathcal{P}$-projective object in $\mathrm{Fun}(I, \mathrm{vect}_K)$. We need to show that it is $\mathcal{P}$-free. Consider a minimal cover $C_0 \to \mathcal{R}M$ in $\mathrm{Fun}(J, \mathrm{vect}_K)$ (see 3.5). Its adjoint $\mathcal{L}C_0 \to M$, according to Theorem 5.14, is then a minimal $\mathcal{P}$-cover of M in $\mathrm{Fun}(I, \mathrm{vect}_K)$, and therefore has to be an isomorphism since M is $\mathcal{P}$-projective. Note that $\mathcal{L}C_0$ is $\mathcal{P}$-free, and consequently so is M. □

To summarize, if the functor $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is thin, then the collection $\mathcal{P}$ is independent (see 5.13) and acyclic (see 5.16.(2)), and every functor $M\colon I \to \mathrm{vect}_K$ has a minimal $\mathcal{P}$-resolution (see 5.16.(1)). Thus thinness guarantees that the $\mathcal{P}$-Betti diagram $\beta^d_{\mathcal{P}}M\colon \mathcal{P} \to \mathbb{N}$ is well defined for every $d \geq 0$ (see 2.6). The rest of this section is devoted to describing methods to calculate some values of these $\mathcal{P}$-Betti diagrams. We start by looking at what we can prove for a general poset J:

5.17 Proposition *Let I, J be finite posets, and $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ and $M\colon I \to \mathrm{vect}_K$ be functors.*

(1) *If $\mathcal{T}$ is thin, then $\beta^0_{\mathcal{P}}M(\mathcal{T}(a)) = \beta^0(\mathcal{R}M)(a)$ for every a in J for which $\mathcal{T}(a) \neq 0$.*
(2) *If $\mathcal{T}$ is flat, then $\beta^d_{\mathcal{P}}M(\mathcal{T}(a)) = \beta^d(\mathcal{R}M)(a)$ for every $d \geq 0$ and every a in J for which $\mathcal{T}(a) \neq 0$.*

Proof Let $f\colon C_0 \to \mathcal{R}M$ be a minimal cover. By definition of $\beta^0(\mathcal{R}M)$, the functor C_0 is isomorphic to $\bigoplus_{a\in J} K(a, -)^{\beta^0(\mathcal{R}M)(a)}$. Recall that, for every a in J, the functor $\mathcal{L}K(a, -)$ is isomorphic to $\mathcal{T}(a)$. This, together with the fact that $\mathcal{L}$ commutes with direct sums, implies that $\mathcal{L}C_0$ is isomorphic to the functor

$$\bigoplus_{\substack{a\in J\\ \mathcal{T}(a)\neq 0}} \mathcal{T}(a)^{\beta^0(\mathcal{R}M)(a)}.$$

Suppose $\mathcal{T}$ is thin. According to Theorem 5.14, the minimal $\mathcal{P}$-cover of M is given by the adjoint $g\colon \mathcal{L}C_0 \to M$ of the minimal cover $f\colon C_0 \to \mathcal{R}M$. We conclude that the equality $\beta^0_{\mathcal{P}}M(\mathcal{T}(a)) = \beta^0(\mathcal{R}M)(a)$ holds for all a for which $\mathcal{T}(a) \neq 0$. This shows (1).

Suppose $\mathcal{T}$ is flat. The natural transformations $\ker(g) \hookrightarrow \mathcal{L}C_0$ and $g\colon \mathcal{L}C_0 \to M$ form a $\mathcal{P}$-exact sequence in $\mathrm{Fun}(I, \mathrm{vect}_K)$. According to Proposition 5.6, after applying $\mathcal{R}$, the natural transformations $\mathcal{R}(\ker(g) \hookrightarrow \mathcal{L}C_0)$ and $\mathcal{R}g$ form therefore an exact sequence in $\mathrm{Fun}(J, \mathrm{vect}_K)$. Since $\mathcal{R}$ is right adjoint, it preserves monomorphisms and hence $\mathcal{R}\ker(g)$ is the kernel of $\mathcal{R}g$. We can then form the following commutative diagram with the indicated arrows representing epimorphisms and monomorphisms, and with the vertical sequences being exact:

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, hook] & \mathcal{R}\ker(g) \arrow[d, hook] \\
C_0 \arrow[r, "\eta"] \arrow[d, two heads, "f"'] & \mathcal{R}\mathcal{L}C_0 \arrow[d, two heads, "\mathcal{R}g"] \\
\mathcal{R}M \arrow[r, equal] & \mathcal{R}M
\end{tikzcd}$$

By flatness, $\mathcal{R}\mathcal{L}C_0$ is free and hence, by minimality of f, the unit η is an isomorphism. The natural transformation represented by the top horizontal arrow in this diagram is then also an isomorphism. Consequently, for every $d \geq 0$ and every a in J, $\beta^d(\ker(f))(a) = \beta^d(\mathcal{R}\ker(g))(a)$, and we conclude with the following sequence of equalities:

$$\begin{aligned}
\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) &= \beta^{d-1}_{\mathcal{P}}(\ker(g))(\mathcal{T}(a)) && \text{(Since } g\colon \mathcal{L}C_0 \to M \text{ is a minimal } \mathcal{P}-\text{cover)} \\
&= \beta^{d-1}(\mathcal{R}\ker(g))(a) && \text{(By induction)} \\
&= \beta^{d-1}(\ker(f))(a) && \text{(By isomorphism)} \\
&= \beta^d(\mathcal{R}M)(a). && \text{(Since } f\colon C_0 \to \mathcal{R}M \text{ is a minimal cover)}
\end{aligned}$$

This proves (2). □

Proposition 5.17.(1) provides an algorithm to calculate the 0^{th} $\mathcal{P}$-Betti diagram of a functor $M\colon I \to \text{vect}_K$ when $\mathcal{T}$ is thin: first take $\mathcal{R}M\colon J \to \text{vect}_K$, then calculate its standard 0^{th} Betti diagram (over the poset J), and finally restrict the obtained 0^{th} Betti diagram to the a in J for which $\mathcal{T}(a) \neq 0$. The standard 0^{th} Betti diagrams can be calculated for example using radicals (see 3.5). This algorithm can be then used inductively in order to calculate the $\mathcal{P}$-Betti diagrams $\beta^d_{\mathcal{P}} M$ of M for all $d \geq 0$, as explained in 2.6. In every step of this procedure we need to evaluate the functor $\mathcal{R}$ on successive kernels (see the second step in the sequence of equalities at the end of the proof of 5.17). We would like to avoid that step, for example by showing beforehand that the $\mathcal{P}$-Betti diagrams over I at $\mathcal{T}(a)$ are equal to the corresponding Betti diagrams over J at a. This is the case if, for example, $\mathcal{T}$ is flat (see Proposition 5.17.(2)). However, our key examples (see Sect. 6) are not flat but thin. For an arbitrary thin functor there could be elements a in J for which the numbers $\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) \neq \beta^d(\mathcal{R}M)(a)$ differ. Such elements are called $\mathcal{T}$-degenerate:

5.18 Degeneracy locus Let $\mathcal{T}\colon J^{\text{op}} \to \text{Fun}(I, \text{vect}_K)$ be thin. An element a in J is called $\mathcal{T}$-**degenerate** if $\mathcal{T}(a) = 0$ or if $\mathcal{T}(a) \neq 0$ and there exists a functor $M\colon I \to \text{vect}_K$ and a natural number $d \geq 0$, for which $\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) \neq \beta^d(\mathcal{R}M)(a)$. The collection of $\mathcal{T}$-degenerate elements is called **degeneracy locus** of $\mathcal{T}$.

If $\mathcal{T}$ is flat, then its degeneracy locus is given by $\{a \in J \mid \mathcal{T}(a) = 0\}$ (see 5.17.(2)). For an arbitrary thin functor $\mathcal{T}$ we do not have an explicit description of its degeneracy locus. However, we can approximate it when J is an upper semilattice.

5.19 Theorem *[Main result, first half] Let I be a finite poset, J a finite upper semilattice, and $\mathcal{T}\colon J^{\text{op}} \to \text{Fun}(I, \text{vect}_K)$ a thin functor. Then the degeneracy locus of $\mathcal{T}$*

(see 5.18) *is contained in the set*

$$\bigcup_{\substack{a\in J\\ d\geq 0}} \operatorname{supp}(\beta^d \ker \eta_a) \subseteq J.$$

Proof Write $\mathcal{G} := \bigcup_{a,d} \operatorname{supp}(\beta^d \ker \eta_a)$. Let us prove the contraposition, that for $M\colon I \to \mathrm{vect}_K$ a functor, a in $J\backslash\mathcal{G}$, and $d \geq 0$, we have $\mathcal{T}(a) \neq 0$ and $\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) = \beta^d(\mathcal{R}M)(a)$. Note that if $\mathcal{T}(a) = 0$, then $\operatorname{supp}(\beta^0 \ker \eta_a) = \{a\}$, so the first statement is true. We proceed by induction on $d \geq 0$ for the second statement. The case $d = 0$ is the statement of Proposition 5.17. Now let $d > 0$.

We proceed exactly as in the proof of Proposition 5.17. Let $M\colon I \to \mathrm{vect}_K$ be a functor and consider the following commutative diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, hook] & \mathcal{R}\ker(g) \arrow[d, hook] \\
C_0 \arrow[r, "\eta"] \arrow[d, two heads, "f"'] & \mathcal{RL}C_0 \arrow[d, two heads, "\mathcal{R}g"] \\
\mathcal{R}M \arrow[r, equal] & \mathcal{R}M
\end{tikzcd}$$

where f is a minimal cover in $\mathrm{Fun}(J, \mathrm{vect}_K)$, g is its left adjoint, which according to Theorem 5.14 is a minimal $\mathcal{P}$-cover in $\mathrm{Fun}(I, \mathrm{vect}_K)$, the indicated arrows are epimorphisms and monomorphisms, and the vertical sequences are exact. The thinness assumption implies that the unit natural transformation η (the middle horizontal arrow in the above diagram) is an epimorphism. This, together with the exactness of the vertical sequences, implies that the natural transformation represented by the top horizontal arrow is also an epimorphism. Also by exactness of the vertical sequences, the kernel of this natural transformation is isomorphic to $\ker(\eta)$. In this way we obtain an exact sequence in $\mathrm{Fun}(J, \mathrm{vect}_K)$ of the form

$$0 \longrightarrow \ker(\eta) \longrightarrow \ker(f) \longrightarrow \mathcal{R}\ker(g) \longrightarrow 0.$$

Let us show that for all a in $J\backslash\mathcal{G}$ and $d \geq 0$, $\beta^d(\ker(f))(a) = \beta^d(\mathcal{R}\ker(g))(a)$. Since C_0 is free and both $\mathcal{R}$ and $\mathcal{L}$ commute with direct sums, $\ker(\eta)$ is isomorphic to a direct sum of functors of the form $\ker(\eta_a\colon K(a,-) \to \mathcal{RL}K(a,-))$. By definition of $\mathcal{G}$, we have the containment $\operatorname{supp}(\beta^d \ker(\eta)) \subseteq \mathcal{G}$ for every $d \geq 0$. Thus, for every $d \geq 0$ and a in $J \setminus \mathcal{G}$, $\beta^d(\ker(\eta))(a) = 0$. Since J is an upper semilattice by assumption, we apply Corollary 3.9 to deduce that $\beta^d(\ker(f))(a) = \beta^d(\mathcal{R}\ker(g))(a)$.

We conclude with the following sequence of equalities:

$$\begin{aligned}
\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) &= \beta^{d-1}_{\mathcal{P}}(\ker(g))(\mathcal{T}(a)) && \text{(Since } g\colon \mathcal{L}C_0 \to M \text{ is a minimal } \mathcal{P}\text{-cover)}\\
&= \beta^{d-1}(\mathcal{R}\ker(g))(a) && \text{(By induction)}\\
&= \beta^{d-1}(\ker(f))(a) && \text{(By our previous claim)}
\end{aligned}$$

$$= \beta^d(\mathcal{R}M)(a). \qquad \text{(Since } f\colon C_0 \to \mathcal{R}M \text{ is a minimal cover)}$$

□

With some additional hypotheses on the parameterization, we get an exact description of the degeneracy locus.

5.20 Corollary *[Main result, second half] Let I be a finite poset, J a finite upper semilattice, and $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ a thin functor. Suppose that, for all a in J, the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$ is contained in $\{b \in J \mid \mathcal{T}(b) = 0\}$. Then the degeneracy locus of $\mathcal{T}$ is $\{b \in J \mid \mathcal{T}(b) = 0\}$, which is equal to $\cup_{a\in J} \langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$.*

Explicitly, for every functor $M\colon I \to \mathrm{vect}_K$, every element a in J such that $\mathcal{T}(a) \neq 0$, and every $d \geq 0$,

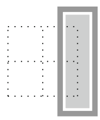
$$\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) = \dim H_d(\mathcal{K}_a \mathcal{R}M).$$

Proof For a in J and $d \geq 0$, $\mathrm{supp}(\beta^d \ker \eta_a)$ is contained in $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$, which itself is contained in $\{b \in J \mid \mathcal{T}(b) = 0\}$ by hypothesis. By Theorem 5.19, we deduce that the degeneracy locus of $\mathcal{T}$ is also contained in $\{b \in J \mid \mathcal{T}(b) = 0\}$. In fact, they are equal, and they are both equal to $\cup_{a\in J} \langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$. By definition of the degeneracy locus and Theorem 3.8 respectively, for all $M\colon I \to \mathrm{vect}_K$, a in J such that $\mathcal{T}(a) \neq 0$, and $d \geq 0$, we get the equalities

$$\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) = \beta^d \mathcal{R}M(a) = \dim H_d(\mathcal{K}_a \mathcal{R}M).$$

□

5.21 Example Our running Example 5.1 satisfies the conditions of Corollary 5.20. This is proved in general for lower hooks in Example F, but here we can check this by hand: for instance, for $a = (00, 10)$, we computed the unit natural transformation η_a in 5.10. We find that

$\ker \eta_a =$  ,

and so $\mathrm{supp}(\beta^0 \ker \eta_a) = \{(10, 10)\} = \langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$, and from 5.1 we know that $\mathcal{T}((10, 10)) = 0$. We reason similarly for all other a in J.

6 Examples

This last section is devoted to examples of independent and acyclic collections together with calculations of the induced relative Betti diagrams in the category $\mathrm{Fun}(I, \mathrm{vect}_K)$, where $(I, \leq)$ is a finite poset. Our strategy for performing these calculations is to find an appropriate parameterization of the collection via an upper semilattice J and a functor $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ as explained in Sect. 5.

Example A. Singleton

We start with the simplest nonempty example. Choose a functor $P_0 : I \to \mathrm{vect}_K$ and consider the collection $\{P_0\}$ consisting only of this chosen functor. The simplest way to parameterize this collection is to consider the poset [0] with a single element 0 and the functor $\mathcal{T} : [0]^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ sending 0 to P_0.

We do a case analysis by the dimension of $\mathrm{Nat}_I(P_0, P_0)$. If $\dim \mathrm{Nat}_I(P_0, P_0) = 0$, then $P_0 = 0$ and the collection $\mathcal{P} = \{\mathcal{T}(a) \mid a \in [0],\ \mathcal{T}(a) \neq 0\}$ is empty. For every functor $M : I \to \mathrm{vect}_K$, $0 \to M$ is a minimal $\mathcal{P}$-resolution, and the Betti diagrams of M are therefore given by the empty function $\varnothing \to \mathbb{N}$.

If $\dim \mathrm{Nat}_I(P_0, P_0) = 1$, then the functor $\mathcal{T}$ is flat (explicitly, $K(0, -) \to \mathrm{Nat}_I(P_0, P_0)$ is an isomorphism). For every functor $M : I \to \mathrm{vect}_K$, the minimal cover of $\mathcal{R}M = \mathrm{Nat}_I(P_0, M)$ is the identity, so by Theorem 5.14, the minimal $\mathcal{P}$-cover of M is its left adjoint, which is the evaluation natural transformation $P_0 \otimes_K \mathrm{Nat}_I(P_0, M) \to M$. By Proposition 5.17:

$$\beta^d_{\mathcal{P}} M(P_0) = \begin{cases} \dim \mathrm{Nat}_I(P_0, M) & \text{if } d = 0, \\ 0 & \text{if } d > 0. \end{cases}$$

If $\dim \mathrm{Nat}_I(P_0, P_0) > 1$, then the functor $\mathcal{T}$ is not thin, and we observe for instance that the minimal cover of $\mathcal{R}P_0$ is adjoint to the $\mathcal{P}$-cover $P_0 \otimes_K \mathrm{Nat}_I(P_0, P_0) \to P_0$, which is not the minimal $\mathcal{P}$-cover of P_0.

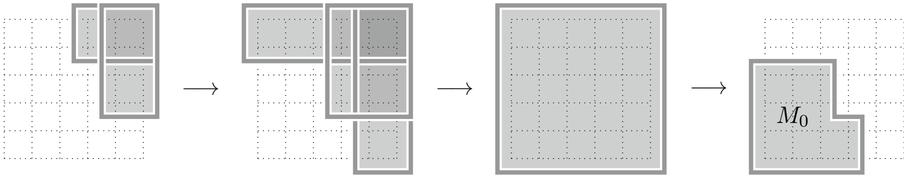
We now proceed with the remaining examples, studying homological algebra relative to various collections of functors in $\mathrm{Fun}(I, \mathrm{vect}_K)$. In each example, to the extent that is possible, we proceed as follows. We start by defining a **collection** and its **parameterization** by a functor $\mathcal{T} : J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$. We then check if the functor $\mathcal{T}$ is flat or thin.

- If $\mathcal{T}$ is **flat**, then we can apply Proposition 5.17.(2).
- If $\mathcal{T}$ is **thin**, then our aim is to apply Corollary 5.20. To do this we need to verify the other conditions of the corollary, namely that the poset J is an **upper semilattice** and that $\mathcal{T}$ sends the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a) \rangle$ to 0 for all a in J. We call this latter assumption the **degeneracy** condition.

In both cases, we conclude that we can compute Betti diagrams relative to $\mathcal{P}$ from Koszul complexes over J. To do this effectively, we characterize **parents** in J and then give an explicit formula for **Koszul complexes**. Then, for every functor $M : I \to \mathrm{vect}_K$, element a in J such that $\mathcal{T}(a) \neq 0$, and $d \geq 0$, we have the equality

$$\beta^d_{\mathcal{P}} M(\mathcal{T}(a)) = \dim H_d(\mathcal{K}_a \mathcal{R} M).$$

Finally, we compute an explicit relative projective resolution of the following functor M_0 on a 6×6 grid, whose minimal standard resolution is illustrated below:



As in Example 5.1, the functor M_0 illustrated here is equal to K on its support (vertices in the shaded area) and 0 elsewhere, with all the transition functions $M_0(v \leq w)$ having maximal rank.

Example B. All Subfunctors

Collection

We consider the collection of all nonzero elements in $\mathrm{Sub}(I)$, the set of all subfunctors of the constant functor K_I (see Sect. 4).

Parameterization

Recall that there is a poset relation $\preccurlyeq$ on $\mathrm{Sub}(I)$ (see 4.6) such that $F \preccurlyeq G$ if, and only if, $\mathrm{supp}(F) \supseteq \mathrm{supp}(G)$, which is equivalent to $\mathrm{supp}(F) \supseteq \mathrm{supp}(\beta^0 G)$. According to 4.5 and 4.6, the poset $(\mathrm{Sub}(I), \preccurlyeq)$ can be identified with either the opposite of the inclusion poset $(\mathrm{Up}(I), \subseteq)$ of upsets of I, or with the poset $(\mathrm{Anti}(I), \preccurlyeq)$ of antichains in I. In particular $(\mathrm{Sub}(I), \preccurlyeq)$ is an upper semilattice.

Define $\mathcal{T} \colon (\mathrm{Sub}(I), \preccurlyeq)^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ to be the identity on objects, and to map a relation $F \preccurlyeq G$ to the corresponding natural transformation $G \subseteq F$. The collection $\mathcal{P} = \{\mathcal{T}(F) \mid F \in \mathrm{Sub}(I),\ \mathcal{T}(F) = F \neq 0\}$ consists of all nonzero subfunctors of K_I.

Flatness

This requires the additional assumption of I having a unique maximal element.

6.1 Proposition *Let I be a finite poset with a unique maximal element. Then the parameterization $\mathcal{T}$ is flat.*

Proof Let $F, G \subseteq K_I$ be nonzero subfunctors. By [19, Prop. 3.10(2)], the dimension of the space $\mathrm{Nat}(G, F)$ is equal to the number of connected components of $\mathrm{supp}(G)$ fully contained in $\mathrm{supp}(F)$. By hypothesis of I having a unique maximal element, $\mathrm{supp}(F)$ and $\mathrm{supp}(G)$ each have only one connected component, so $\dim(\mathrm{Nat}(G, F)) = 1$ if $F \preccurlyeq$

G and 0 otherwise. We deduce that the unit natural transformation $\eta_F\colon K(F,-) \to \mathcal{R}\mathcal{L}K(F,-)$ is an isomorphism, and so we conclude that $\mathcal{T}$ is flat. $\square$

According to Proposition 5.17.(2), the flatness of $\mathcal{T}$ gives the equality $\beta^d_{\mathcal{P}}M(F) = \beta^d(\mathcal{R}M)(F)$ for every functor $M\colon I \to \mathrm{vect}_K$, nonzero subfunctor $F \subseteq K_I$, and $d \geq 0$. The $\mathcal{P}$-Betti diagrams over I can be therefore expressed as the standard Betti diagrams over the upper semilattice $(\mathrm{Sub}(I), \preccurlyeq)$. Thus these standard Betti diagrams can be calculated using Koszul complexes, once we have identified and enumerated parents and their meets in $(\mathrm{Sub}(I), \preccurlyeq)$ (see 5.20).

Parents

We identify $(\mathrm{Sub}(I), \preccurlyeq)$ with the opposite poset of upsets $\mathrm{Up}(I)^{\mathrm{op}}$ via the map $F \mapsto \mathrm{supp}(F)$ (see 4.5). For a subset S of I, denote by S^c its complement in I and by $\mathrm{Max}(S)$ the set of maximal elements of S. The sets of parents in $\mathrm{Up}(I)^{\mathrm{op}}$ can be described as follows:

6.2 Lemma *The set of parents of U in* $\mathrm{Up}(I)^{\mathrm{op}}$ *is* $\{U \cup \{v\} \mid v \in \mathrm{Max}(U^c)\}$.

Proof Let v be an element of $\mathrm{Max}(U^c)$. The set $U \cup \{v\}$ is indeed an upset because every element greater than v is not in U^c, and hence it is in U. Since $U \cup \{v\}$ contains U, the former is indeed a lower bound of the latter in $\mathrm{Up}(I)^{\mathrm{op}}$. Finally, since there is only a one-element difference, $U \cup \{v\}$ is a parent of U.

Conversely, let V be a parent of U in $\mathrm{Up}(I)^{\mathrm{op}}$. Then U is properly contained in V. Let v be a maximal element in $V \backslash U$. Since V is an upset, v is also an element of $\mathrm{Max}(U^c)$, and consequent $V = U \cup \{v\}$. $\square$

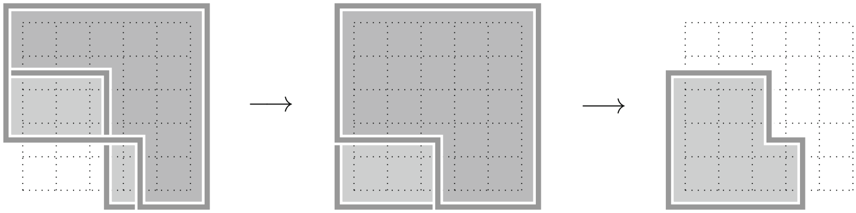
Recall that for an upset U of I, the symbol K_U denotes the unique subfunctor of K_I for which $\mathrm{supp}(K_U) = U$. Thus, in terms of subfunctors, according to Lemma 6.2, the parents of $F \subseteq K_I$ in the poset $(\mathrm{Sub}(I), \preccurlyeq)$ are subfunctors of the form $K_{\mathrm{supp}(F)\cup\{v\}} \subseteq K_I$ for v in $\mathrm{Max}(\mathrm{supp}(F)^c)$.

Koszul Complexes

For $M\colon I \to \mathrm{vect}_K$ and $F \subseteq K_I$, the Koszul complex of $\mathcal{R}M\colon \mathrm{Sub}(I) \to \mathrm{vect}_K$ at F is therefore given in degree $d \geq 0$ by

$$(\mathcal{K}_F\mathcal{R}M)_d = \bigoplus_{\substack{S\subseteq \mathrm{Max}(\mathrm{supp}(F)^c)\\ |S|=d}} \mathrm{Nat}(K_{\mathrm{supp}(F)\cup S}, M),$$

with the differentials as defined in 3.7. The value $\beta^d_{\mathcal{P}}M(F)$ is then given by the dimension of the d^{th} homology of this chain complex. For example, in the case I is the 6×6 grid, the relative projective resolution of M_0 can be illustrated as



Since $\mathcal{T}\colon \mathrm{Sub}(I)^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is flat, so are all of its restrictions (see 5.12). Thus one way of producing examples of flat functors for which we can use Koszul complexes to calculate the associated relative Betti diagrams is to find subposets of $\mathrm{Sub}(I)$ which are upper semilattices (not necessarily sublattices of $(\mathrm{Sub}(I), \preccurlyeq)$) and for which we can identify parents of its elements and meets of subsets of parents that are bounded below. The next example is an illustration of this strategy.

Example C. Translated Functors

Collection

Suppose the poset $(I, \leq)$ is the product $\{0 < \cdots < n\}^r$, which is a distributive lattice. Its elements will be denoted as words $w_1 \cdots w_r$. Fix an antichain S in $\mathrm{Anti}(I)$ and define $I - S := \{v \in I \mid S + v \subseteq I\}$. The set $I - S$ consists of the elements v in I for which all the coordinates of $t + v$ are bounded by n for every s in S. Thus

$$I - S = \prod_{i=1}^{r} \{0, \ldots, n - \max\{s_i \mid s \in S\}\} \ \subseteq\ \{0, \ldots, n\}^r = I.$$

Moreover, the subposet $(I - S) \subseteq I$ is a sublattice.

For v in $I - S$, the subset $S + v \subseteq I$ is also an antichain in I. The associated subfunctor $K(S+v, -) \subseteq K_I$ is called the **translation** of $K(S, -) \subseteq K_I$ by v (see 4.6 for the notation). In this example we consider the collection of all such translations $\mathcal{P} := \{K(S + v, -) \mid v \in I - S\}$.

Parameterization

The lattice $I - S$ provides a natural choice for parameterizing this collection because, for v and w in $I - S$, the inclusion $K(S + v, -) \subseteq K(S + w, -)$ exists if, and only if, $v \geq w$ in I. In particular, $K(S + v, -) = K(S + w, -)$ if, and only if, $v = w$. Set $\mathcal{T}\colon (I - S)^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ to be the functor which maps v in $(I - S)^{\mathrm{op}}$ to the translation $K(S + v, -)$.

Flatness

This functor is flat by Proposition 5.12, since it can be identified with the restriction of the functor $\mathcal{T}$ discussed in Example B to the subposet $\{K(S + v, -) \mid v \in I - S\} \subseteq$

Sub(I). Although the inclusion of posets may fail to be a sublattice inclusion, the poset $I - S$ is a lattice (since it is a sublattice of I) and consequently the $\mathcal{P}$-Betti diagrams can be calculated via Koszul complexes.

Parents

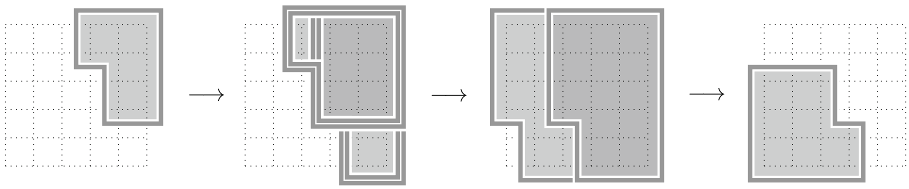
For every element v of $(I - S) \subseteq I$, the parents of v in I are also in $I - S$. Thus the parents of v in $I - S$ are the same as those of v in I.

Koszul Complexes

For $M: I \to \mathrm{vect}_K$ and v in $I - S$, the Koszul complex of $\mathcal{R}M: (I - S) \to \mathrm{vect}_K$ at v is given in degree $d \geq 0$ by

$$(\mathcal{K}_v \mathcal{R}M)_d = \bigoplus_{\substack{T \subseteq \mathcal{U}_{I-S}(v) \\ |T|=d}} \mathrm{Nat}(K(S + \textstyle\bigwedge_{(I-S)} T, -), M),$$

with differentials as defined in 3.7. Finally, the value $\beta^d_{\mathcal{P}} M(K(S + v, -))$ is given by the dimension of the d^{th} homology of this chain complex. For example, if we choose $I = \{0 < \cdots < 5\}^2$ and $S = \{02, 10\} \subset I$, then the projective resolution of M_0 relative to translations of the subfunctor $K(\{02, 10\}, -) \subseteq K_I$ can be illustrated as



where the double lines indicate summands with multiplicity 2.

Example D. Spread Modules

Collection

We consider the following collection:

$$\mathcal{C} := \{\mathrm{coker}(G \subseteq F) \mid G \subseteq F \subseteq K_I\}.$$

Functors in this collection have been studied in other works. They are exactly the spread modules as defined in [5]. Recall that spread modules are defined using two antichains S and T in I such that, for all s in S there exists t in T such that $s \leq t$, and for all t in T there exists s in S such that $s \leq t$. The **spread** with **sources** S and **sinks**

T is the subset of I

$$[S, T] := \{v \in I \mid \exists s \in S, \exists t \in T, s \leq v \leq t\}.$$

The **spread module** with sources S and sinks T is then defined as the functor $K_{[S,T]}\colon I \to \mathrm{vect}_K$ where, for all $v \leq w$ in I,

$$K_{[S,T]}(v) = \begin{cases} K & \text{if } v \in [S,T], \\ 0 & \text{otherwise}, \end{cases} \qquad K_{[S,T]}(v \leq w) = \begin{cases} \mathrm{id}_K & \text{if } v, w \in [S,T], \\ 0 & \text{otherwise}. \end{cases}$$

6.3 Lemma *The following statements about $M\colon I \to \mathrm{vect}_K$ are equivalent:*

(1) *M is isomorphic to a spread module;*
(2) *M is isomorphic to* $\mathrm{coker}(G \subseteq F)$ *for some $G \subseteq F \subseteq K_I$;*
(3) *For all $v \leq w$ in I,* $\dim(M(v)) \leq 1$ *and $M(v \leq w)$ is of maximal rank.*

Proof $(1 \Rightarrow 2)$ If M is isomorphic to a spread module $K_{[S,T]}$, then there is a surjection $\alpha\colon K(S, -) \to M$, and hence M is isomorphic to $\mathrm{coker}(\ker(\alpha) \subseteq K(S, -))$.
$(2 \Rightarrow 3)$ Let $v \leq w$ be elements of I. Since F surjects onto $\mathrm{coker}(G \subseteq F)$, we have $\dim(M(v)) \leq 1$. If $M(v) = 0$ or $M(w) = 0$, then $M(v \leq w)$ is of full rank. Otherwise, v and w are outside of the support of G, which is an upset, and so $M(x) \cong F(x)$ for all $v \leq x \leq w$, and so $M(v \leq w)$ is of full rank.
$(3 \Rightarrow 1)$ Let S be the minimal elements and T the maximal elements of $\mathrm{supp}(M)$, and consider the spread module $K_{[S,T]}$. Note that, for $v \leq w$ in I, if $M(v \leq w)$ is nonzero, then it is an isomorphism of 1-dimensional vector spaces. Thus, for any two elements v, w in a connected component of $\mathrm{supp}(M)$, there exists an isomorphism $\varphi_{v,w}\colon M(v) \to M(w)$ that commutes with the transition maps of M. This isomorphism can be defined as a zigzag composition of transition maps.

Let $C_1, \ldots, C_k$ be the connected components of $\mathrm{supp}(M)$. For each i, fix an element $v_i \in C_i$, and define a natural transformation $f\colon M \to K_{[S,T]}$ by

$$f(v) = \begin{cases} \varphi_{v_i,v}(1)\mathrm{id}_K & \text{if } v \in C_i, i \in \{1, \ldots, k\}, \\ 0 & \text{otherwise}, \end{cases}$$

We conclude by observing that f is an isomorphism. □

Parameterization

Consider the following poset equipped with the product order:

$$\Omega := \{(F, G) \in \mathrm{Sub}(I)^2 \mid F \preccurlyeq G\}$$

The functor $\mathcal{Q}\colon \Omega^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ which sends (F, G) to $\mathrm{coker}(G \subseteq F)$ is a natural parameterization of the collection $\mathcal{C}$.

Non-Thinness

The functor $\mathcal{Q}$ in general is not thin, as the following proposition illustrates, and hence we cannot apply Corollary 5.20.

6.4 Proposition *Suppose I is a finite poset containing two incomparable elements v and w. Then the collection of nonzero elements in $\mathcal{C}$ is not independent and $\mathcal{Q}\colon \Omega^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ is not thin.*

Proof Let $F, G, H\colon I \to \mathrm{vect}_K$ be the unique functors in $\mathcal{C}$ whose supports are equal to the following spreads: $\mathrm{supp}(F) = \{v\}$, $\mathrm{supp}(G) = \{w\}$, and $\mathrm{supp}(H) = \{v, w\}$. Since H is isomorphic to $F \oplus G$, the collection of nonzero elements in $\mathcal{C}$ cannot be independent. Consequently $\mathcal{Q}$ cannot be thin (see Proposition 5.13). $\square$

Although $\mathcal{Q}$ is not thin, it can be used to construct thin functors. Our strategy is to use Proposition 5.9 to look for thin restrictions of $\mathcal{Q}$ to subposets of Ω.

Example E. Single-Source Spread Modules

Collection

We consider the following subcollection of $\mathcal{C}$ (see Example D):

$$\mathcal{C}_0 := \{F \in \mathcal{C} \mid F \text{ has exactly one generator}\}.$$

Note that all the functors in $\mathcal{C}_0$ are of the form $\mathrm{coker}(G \subseteq K(v, -))$ for some v in I. Thus functors in $\mathcal{C}_0$ are isomorphic to spread modules $K_{[S,T]}$ with $|S| = 1$. These spread modules are studied in [5] and are called **single-source spread modules**.

Parameterization

The restriction of $\mathcal{Q}\colon \Omega^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ (see Example D) to the following subposet of Ω is a natural parameterization of the collection $\mathcal{C}_0$:

$$\Omega_0 := \{(K(v, -), G) \in \mathrm{Sub}(I)^2 \mid K(v, -) \preccurlyeq G \text{ and } v \text{ in } I\}.$$

Let $\mathcal{Q}_0\colon \Omega_0^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ denote this restriction.

Thinness

The parameterization $\mathcal{Q}_0$ satisfies the conditions of Proposition 5.9. Indeed, by definition, the functor $\mathcal{Q}_0(K(v, -), G) = \mathrm{coker}(G \subseteq K(v, -))$ has at most one generator, and if the functor is nonzero, then its generator is at v. Moreover, we can check that if $(K(v, -), G) \not\preccurlyeq (K(v', -), G')$ in Ω_0, then $\mathrm{Nat}(\mathcal{Q}_0(K(v', -), G'), \mathcal{Q}_0(K(v, -), G))$ $= 0$. Consequently, $\mathcal{Q}_0$ is thin and we can utilize Proposition 5.17.(1) to express the 0^{th}

$\mathcal{C}_0$-Betti diagrams of functors indexed by I in terms of the standard 0^{th} Betti diagrams of functors indexed by Ω_0.

In order to apply Corollary 5.20 to use Koszul complexes to calculate the higher $\mathcal{C}_0$-Betti diagrams, we need to check that the poset Ω_0 is an upper semilattice and that the functor $\mathcal{Q}_0$ sends $\langle \text{supp}(\beta^0 \ker \eta_a)\rangle$ to 0 for all a in Ω_0.

Upper Semilattice

Suppose $(I, \leq)$ is an upper semilattice. Then Ω_0 is also an upper semilattice. To describe the join operation and the parents of elements in Ω_0, it is convenient to identify it with the following subposet of the product $I \times \text{Up}(I)^{\text{op}}$

$$\Omega_0 \cong \{(v, U) \in I \times \text{Up}(I)^{\text{op}} \mid v \leq u \text{ for every } u \text{ in } U\}$$

by assigning to an element $(K(v, -), G)$ in Ω_0, the pair $(v, \text{supp}(G))$. In this poset $(v_1, U_1) \preccurlyeq (v_2, U_2)$ if and only if $v_1 \leq v_2$ and $U_1 \supseteq U_2$, and hence the join of (v_1, U_1) and (v_2, U_2) in Ω_0 is given by $(v_1 \vee v_2, U_1 \cap U_2)$.

Degeneracy

By definition, for all $a = (v, U)$ in Ω_0, the support of the kernel of the unit $\eta_a \colon K(a, -) \to \mathcal{RL}K(a, -)$ consists of the elements b in Ω_0 such that $b \succcurlyeq a$ and $\mathcal{RL}K(a, b) = \text{Nat}(\mathcal{Q}_0(b), \mathcal{Q}_0(a)) = 0$. Thus the elements in $\text{supp}(\beta^0 \ker \eta_a)$, which are the minimal elements in $\text{supp}(\ker \eta_a)$, are of the form $(u, (u \leq I))$ where u is a minimal element of U. Note further that $\mathcal{Q}_0(v', v' \leq I) = 0$ for every v' in I. Since the collection of elements in Ω_0 of the form $(v', (v' \leq I))$ is closed under joins, we conclude that $\mathcal{Q}_0$ sends every element of $\langle \text{supp}(\beta^0 \ker \eta_a)\rangle$ to 0.

Parents

The poset Ω_0 has the product order and Lemma 6.2 gives the parents of upsets, so the parents of (v, U) in Ω_0 are of the form

$$\mathcal{U}_{\Omega_0}(v, U) = (\mathcal{U}_I(v) \times \{U\}) \,\cup\, \{(v, U \cup \{w\}) \mid w \in (v \leq \text{Max}(U^c))\},$$

where $(v \leq \text{Max}(U^c))$ is the set of maximal elements of U^c bounded below by v.

Koszul Complex

For $M \colon I \to \text{vect}_K$ the value of the functor $\mathcal{R}M \colon \Omega_0 \to \text{vect}_K$ at (v, U) in Ω_0 can be identified with

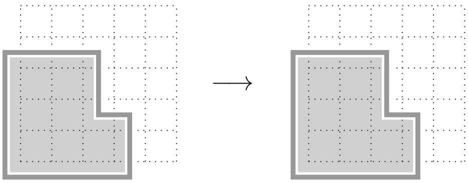
$$\mathcal{R}M(v, U) = \text{Nat}(\text{coker}(K_U \subseteq K(v, -)), M) = \bigcap_{u \in \text{Min}(U)} \ker M(v \leq u),$$

where the second equality comes from identifying every element φ in $\mathrm{Nat}(\mathrm{coker}(K_U \subseteq K(v,-)), M)$ with its value $\varphi(v)(1)$ in $\bigcap_{u \in \mathrm{Min}(U)} \ker M(v \leq u)$.

The Koszul complex of $\mathcal{R}M$ at (v, U) in degree $d \geq 0$ can now be identified with

$$(\mathcal{K}_{(v,U)}\mathcal{R}M)_d = \bigoplus_{\substack{S \subseteq \mathcal{U}_I(v) \\ T \subseteq (v \leq \mathrm{Max}(U^c)) \\ |S|+|T|=d \\ S \text{ has lower bound}}} \bigcap_{u \in \mathrm{Min}(U \cup T)} \ker M\left(\textstyle\bigwedge_{(I \leq v)} S \leq u\right),$$

with differentials as defined in 3.7. Finally, we obtain the relative Betti diagrams of M from the homology of this chain complex. For example, the relative projective resolution of $M_0 = \mathrm{coker}(K_{\{04,32,40\}} \subseteq K(00,-))$ is itself, since it is already a single-source spread module:



Example F. Lower Hooks

Collection

Let $(I, \leq)$ be a finite upper semilattice. We now consider the collection of lower hooks, originally defined in [6], which we define as the subcollection of $\mathcal{C}$ (see Example D)

$$\mathcal{C}_L := \{\mathrm{coker}(K(w,-) \subseteq K(v,-)) \mid K(w,-) \subseteq K(v,-) \subseteq K_I\}.$$

Parameterization

Let $(J, \preccurlyeq)$ be the poset $\{(v, w) \mid v, w \in I, v \leq w\}$ equipped with the product order: $(v, w) \preccurlyeq (v', w')$ in J if and only if $v \leq v'$ and $w \leq w'$ in I. We identify J with the subposet of the poset Ω_0 (see Example E) consisting of the pairs $(v, w \leq I)$ for (v, w) in J. Via this identification, we parameterize lower hooks by the restriction of the functor $\mathcal{Q}_0 \colon \Omega_0^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ (see Example E) to J, and denote it by $\mathcal{T}$. Explicitly, for $a = (v, w)$ in J, the functor $\mathcal{T}(a)$ is given by $\mathrm{coker}(K(w,-) \subseteq K(v,-))$.

Thinness

The functor $\mathcal{T}$ is thin by Proposition 5.12, since it is a restriction of the thin functor $\mathcal{Q}_0$.

Upper Semilattice

The poset J is a finite upper semilattice, inheriting the structure from I. Note however that J is not a sublattice of Ω_0.

Degeneracy

For all $a = (v, w)$ in J, the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle \subseteq J$ is sent to 0 by $\mathcal{T}$. Indeed, the support of the kernel of the unit $\eta_a \colon K(a, -) \to \mathcal{RLK}(a, -)$ consists of the elements b of J such that $b \succcurlyeq a$ and $\mathcal{RLK}(a, b) = \mathrm{Nat}(\mathcal{T}(a), \mathcal{T}(b)) = 0$. This support has a single minimal element, (w, w), which is therefore also the only element of $\mathrm{supp}(\beta^0 \ker \eta_a)$. Thus, in this case, $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle = \{(w, w)\}$, which is sent to 0 by $\mathcal{T}$.

By Corollary 5.20, we conclude that the Betti diagrams relative to lower hooks can be computed via Koszul complexes over J.

Parents

First, for all (v, w) in J, we have

$$\mathcal{U}_J(v, w) = (\mathcal{U}_I(v) \times \{w\}) \cup (\{v\} \times (v \leq \mathcal{U}_I(w))),$$

where $(v \leq \mathcal{U}_I(w))$ is the subset of $\mathcal{U}_I(w)$ of elements greater or equal to v.

Koszul Complexes

Next, we identify $\mathcal{R}M(v, w) = \mathrm{Nat}(\mathcal{T}(v, w), M)$ with $\ker(M(v \leq w))$. Thus we can compute the Koszul complex of $\mathcal{R}M$ at (v, w) in degree $d \geq 0$ as

$$(\mathcal{K}_{(v,w)}\mathcal{R}M)_d = \bigoplus_{\substack{S \subseteq \mathcal{U}_I(v),\ T \subseteq (v \leq \mathcal{U}_I(w)) \\ |S|+|T|=d \\ S \text{ has lower bound}}} \ker M\left(\bigwedge\nolimits_{(I \leq v)} S \leq \bigwedge\nolimits_{(I \leq w)} T\right),$$

with differentials as defined in 3.7. Finally, we obtain the relative Betti diagrams of M from the homology of this chain complex.

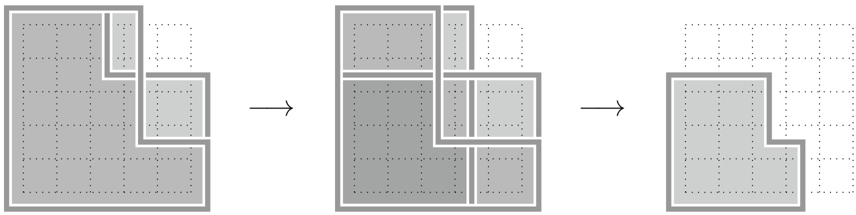
Variation on Lower Hooks

Now consider the subcollection of $\mathcal{C}$

$$\mathcal{C}_\infty := \mathcal{C}_L \cup \{\mathrm{coker}(0 \subseteq K(v, -)) \mid v \in I\}.$$

We parameterize this by the finite upper semilattice $J_\infty = \{(v, w) \mid v, w \in I \cup \{\infty\}, v \leq w\}$, where ∞ is greater than every element of I. The upper semilattice J is a sublattice of J_∞. We extend the functor $\mathcal{T}\colon J^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ to J_∞ by sending (v, ∞) to $K(v, -)$ and denote it by the symbol $\mathcal{T}_\infty$. The conditions for Corollary 5.20 also hold for this extension. In addition, $\mathcal{C}_\infty$-exactness implies exactness and **rank additivity** [6, Prop. 4.3]: given a short $\mathcal{C}_\infty$-exact sequence $0 \to M \to N \to L \to 0$ and $v \leq w$ in I, we have $\operatorname{rank} N(v \leq w) = \operatorname{rank} M(v \leq w) + \operatorname{rank} L(v \leq w)$.

There is a similar formula for Koszul complexes where we identify $\mathcal{R}M(v, \infty)$ with $M(v)$. For both of these examples ($\mathcal{T}$ and $\mathcal{T}_\infty$), the relative projective resolution of M_0 is



As observed above, this sequence is rank additive.

Relative Projective Dimension

By [9, Corollary 10.18] we can deduce an upper bound for the projective dimension of a functor in $\mathrm{Fun}(J, \mathrm{vect}_K)$, i.e. the maximal length of its minimal projective resolution. By Corollary 5.20, the maximal length of a minimal projective resolution in $\mathrm{Fun}(I, \mathrm{vect}_K)$ relative to lower hooks will have the same upper bound. Recent work [7] proposes a tighter bound in the case where I is a finite lattice. Here we present a proof of this result using Koszul complexes.

6.5 Proposition *[cf. [7, Theorem 5.18]] Let I be a finite lattice such that every its element has at most n parents. Then the projective dimension relative to lower hooks of every functor $M\colon I \to \mathrm{vect}_K$ is bounded above by $2n - 2$.*

Proof By [9, Corollary 10.18], every functor M in $\mathrm{Fun}(I, \mathrm{vect}_K)$ has projective dimension at most n. For the same reason and by observing that J is also a lattice where every element has at most $2n$ parents, a functor in $\mathrm{Fun}(J, \mathrm{vect}_K)$ has projective dimension at most $2n$. In particular, this is the case for the functor $\mathcal{R}M\colon J \to \mathrm{vect}_K$.

Let $\pi_1, \pi_2\colon J \to I$ be the functors defined by $\pi_1(v, w) = v$ and $\pi_2(v, w) = w$. Consider the compositions $M\pi_1$ and $M\pi_2$ in $\mathrm{Fun}(J, \mathrm{vect}_K)$ and the natural transformation $\alpha : M\pi_1 \to M\pi_2$ whose component at any (v, w) in J is $M(v \leq w)$. Observe that there is an exact sequence of functors $0 \to \mathcal{R}M = \ker(\alpha) \hookrightarrow M\pi_1 \xrightarrow{\alpha} M\pi_2$. Now let $C \to M$ be a minimal free resolution in $\mathrm{Fun}(I, \mathrm{vect}_K)$. For $i = 1, 2$, the sequence $C\pi_i \to M\pi_i$ in $\mathrm{Fun}(J, \mathrm{vect}_K)$ is also exact because exactness is defined componentwise. Moreover, for u in I and (v, w) in J, the vector space $(K(u, -)\pi_1)(v, w)$ is nonzero if, and only if, $u \leq v$, or equivalently $(u, u) \leq (v, w)$. Thus, $K(u, -)\pi_1$ is isomorphic to the free functor $K((u, u), -)$. As a consequence, the sequence $C\pi_1 \to M\pi_1$ is a free resolution in $\mathrm{Fun}(J, \mathrm{vect}_K)$, and so $M\pi_1$ has projective dimension at most n. Analogously, $K(u, -)\pi_2$ coincides with the free functor

$K((e, u), -)$, where e is the global minimum of the lattice (it exists because I is finite). We have again that the sequence $C\pi_2 \to M\pi_2$ is a free resolution in $\mathrm{Fun}(J, \mathrm{vect}_K)$, and $M\pi_2$ has projective dimension at most n.

Now consider the exact sequence $0 \to \mathcal{R}M \to M\pi_1 \to \mathrm{im}(\alpha) \to 0$ and let (v, w) be an element in J. Since the functor $\mathcal{K}_{(v,w)}$ is exact, the sequence $0 \to \mathcal{K}_{(v,w)}\mathcal{R}M \to \mathcal{K}_{(v,w)}M\pi_1 \to \mathcal{K}_{(v,w)}\mathrm{im}(\alpha) \to 0$ is also exact. As we observed, the elements in the poset J have at most $2n$ parents, so we obtain the following long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_{2n}(\mathcal{K}_{(v,w)}\mathcal{R}M) \arrow[r] & H_{2n}(\mathcal{K}_{(v,w)}M\pi_1) \arrow[r] & H_{2n}(\mathcal{K}_{(v,w)}\mathrm{im}(\alpha)) \arrow[dll, rounded corners] \\
& H_{2n-1}(\mathcal{K}_{(v,w)}\mathcal{R}M) \arrow[r] & H_{2n-1}(\mathcal{K}_{(v,w)}M\pi_1) \arrow[r] & H_{2n-1}(\mathcal{K}_{(v,w)}\mathrm{im}(\alpha)) \arrow[r] & \cdots
\end{tikzcd}$$

However, as seen above the $2n^{\mathrm{th}}$ Betti diagram $\beta^{2n}(M\pi_1) \colon J \to \mathbb{N}$ is identically 0, and hence $H_{2n}(\mathcal{K}_{(v,w)}M\pi_1) = 0$. We deduce the equalities $\beta^{2n}\mathcal{R}M(v, w) = \dim H_{2n}(\mathcal{K}_{(v,w)}\mathcal{R}M) = 0$. Similarly, consider the short exact sequence $0 \to \mathrm{im}(\alpha) \to M\pi_2 \to Q \to 0$ where $Q := M\pi_2/\mathrm{im}(\alpha)$. As before, it leads to a long exact sequence

$$0 \longrightarrow H_{2n}(\mathcal{K}_{(v,w)}\mathrm{im}(\alpha)) \longrightarrow H_{2n}(\mathcal{K}_{(v,w)}M\pi_2) \longrightarrow H_{2n}(\mathcal{K}_{(v,w)}Q) \longrightarrow \cdots$$

Consequently, $H_{2n}(\mathcal{K}_{(v,w)}M\pi_2) = 0$, and so $H_{2n}(\mathcal{K}_{(v,w)}\mathrm{im}(\alpha)) = 0$. Moreover, considering the first long exact sequence, we also have $H_{2n-1}(\mathcal{K}_{(v,w)}\pi_1) = 0$, and so $\beta^{2n-1}\mathcal{R}M(v, w) = \dim H_{2n-1}(\mathcal{K}_{(v,w)}\mathcal{R}M) = 0$.

The functor $\mathcal{R}M$ therefore has projective dimension at most $2n - 2$, and so M has projective dimension relative to lower hooks at most $2n - 2$. □

Example G. Rectangles

We now illustrate an example where the assumptions of Corollary 5.20 fail.

Collection

Let $(I, \leq)$ be a finite upper semilattice. We define the **rectangle from** v **to** w as the spread $[v, w] := \{x \in I \mid v \leq x \leq w\}$ of I (see Example D) and we consider the subcollection of $\mathcal{C}$ of rectangle functors:

$$\mathcal{C}_R := \{K_{[v,w]} \mid v, w \in I,\ v \leq w\}.$$

Parameterization

Let $(J, \preccurlyeq)$ be the poset $\{(v, w) \mid v, w \in I, v \leq w\}$ with the product order, as in the example of lower hooks (see Example F). This time, we identify J with the subposet $\{(v, \ker(K(v, -) \to K(-, w)^*) \mid v \leq w\}$ of Ω_0, where the functor $K(-, w)^*$ is equal to K at u, when $u \leq w$, and 0 otherwise, with transition functions of full rank.

Via this identification, we parameterize rectangle functors by the restriction of the functor $\mathcal{Q}_0 \colon \Omega_0^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ (see Example E) to J, and denote it by $\mathcal{T}$.

Thinness

The functor $\mathcal{T}$ is thin by Proposition 5.12, since it is a restriction of the thin functor $\mathcal{Q}_0$.

Upper Semilattice

The poset J is a finite upper semilattice just as in the previous example.

Problem With Degeneracy Locus

Let I be the finite grid $\{0 < \cdots < 5\}^2$ with the product order and $M_0 \colon I \to \mathrm{vect}_K$ a functor. Choose $a := (04, 24)$ in J. It has has two parents, $(03, 24)$ and $(04, 14)$. The Koszul complex of $\mathcal{R}M_0$ at a is therefore

$$\begin{aligned}\mathrm{Nat}(\mathcal{T}(03, 14), M_0) &\to \mathrm{Nat}(\mathcal{T}(03, 24), M_0) \oplus \mathrm{Nat}(\mathcal{T}(04, 14), M_0)\\ &\to \mathrm{Nat}(\mathcal{T}(04, 24), M_0),\end{aligned}$$

which simplifies to $0 \to \mathrm{Nat}(\mathcal{T}(03, 24), M_0) = K \to 0$, and so $\beta^1 \mathcal{R}M_0(a) = 1$. However, in Example H below we compute, using a different parameterization of the collection $\mathcal{C}_R$, the relative Betti diagrams of M_0. In particular, we determine $\beta^1_{\mathcal{P}} M_0(\mathcal{T}(a)) = 0$, hence $\beta^1_{\mathcal{P}} M_0(\mathcal{T}(a)) \neq \beta^1 \mathcal{R}M_0(a)$. The reason for this non-equality is that $\mathcal{T}$ does not send the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a) \rangle$ to 0 for all a in J. In fact, $\mathcal{T}$ does not send any element of J to 0, but sometimes $\ker \eta_a$ is nonzero, and so the support of $\beta^0 \ker \eta_a$ must be sent to nonzero functors.

Example H. Rectangles on a Grid

We can amend the example of rectangles (see Example G) when we have more control over the poset I. In this example, we consider $I := \{0 < \cdots < n\}^r$ a finite grid with the product order, denoted by $\leq$. For v, w in I, we write $v + (w - v)_i$ for the element $(v_1, \ldots, v_{i-1}, w_i, v_{i+1}, \ldots, v_r)$.

Collection

We consider the same collection $\mathcal{C}_R$ as in Example G.

Parameterization

Let $(J, \preccurlyeq)$ be the poset $\{(v, w) \mid v, w \in I, v \leq w\}$ with the product order, as in the previous example. This time, we identify J with the subposet

$$\left\{\left(v, \bigcup_{i=1}^{r}(v + (w - v)_i \leq I)\right) \,\middle|\, v, w \in I, v \leq w\right\}$$

of Ω_0 (see Example E). Via this identification, we parameterize rectangle functors by the restriction of the functor $\mathcal{T} \colon \Omega_0^{\mathrm{op}} \to \mathrm{Fun}(I, \mathrm{vect}_K)$ (see Example E) to J, and denote it by $\mathcal{T}$. Explicitly, $\mathcal{T}$ maps (v, w) to the functor $\mathrm{coker}(\bigoplus_{i=1}^{r} K(v + (w - v)_i, -) \to K(v, -))$.

Thinness

The functor $\mathcal{T}$ is thin by Proposition 5.12, since it is a restriction of the thin functor $\mathcal{Q}_0$.

Upper Semilattice

The poset J is a finite upper semilattice just as in the previous example.

Degeneracy

For all $a = (v, w)$ in J, $\mathcal{T}$ sends the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$ to 0. Indeed, the support of the kernel of the unit $\eta_a \colon K(a, -) \to \mathcal{RL}K(a, -)$ consists of the elements b in J such that $b \succcurlyeq a$ and $\mathcal{RL}K(a, b) = \mathrm{Nat}(\mathcal{T}(b), \mathcal{T}(a)) = 0$. Thus the elements of $\mathrm{supp}(\beta^0 \ker \eta_a)$, which are the minimal elements of $\mathrm{supp}(\ker \eta_a)$, are of the form $(v+(w-v)_i, w)$ where $i \in \{1, \ldots, r\}$. The sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$ then consists of the elements (v_X, w), for every nonempty subset X of $\{1, \ldots, r\}$, where, for all i in $\{1, \ldots, r\}$, the element v_X in I is defined by

$$(v_X)_i := \begin{cases} w_i & \text{if } i \in X, \\ v_i & \text{otherwise.} \end{cases}$$

Note that for (v', w') in J, $\mathcal{T}(v', w') = 0$ if, and only if, there exists $i \in \{1, \ldots, r\}$ such that $v'_i = w'_i$. As a consequence, $\mathcal{T}$ sends the elements of the sublattice $\langle \mathrm{supp}(\beta^0 \ker \eta_a)\rangle$ to 0.

By Corollary 5.20, we can compute Betti diagrams relative to rectangles via Koszul complexes.

Parents

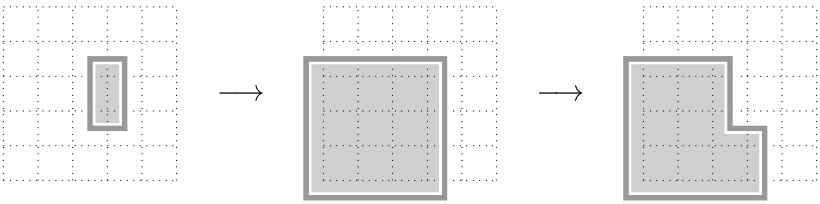
Parents are exactly the same as for lower hooks (see Example F).

Koszul Complexes

For (v, w) in J, we can identify $\mathcal{R}M(v, w) = \mathrm{Nat}(\mathcal{T}(v, w), M)$ with $\bigcap_{i=1}^{r} \ker M(v \leq v + (w - v)_i)$. Thus we can compute the Koszul complex of $\mathcal{R}M$ at (v, w) in degree $d \geq 0$ as

$$(\mathcal{K}_{(v,w)}\mathcal{R}M)_d = \bigoplus_{\substack{S \subseteq \mathcal{U}_I(v) \\ T \subseteq (v \leq \mathcal{U}_I(w)) \\ |S|+|T|=d}} \bigcap_{i=1}^{r} \ker M\left(v_S \leq v_S + (w_T - v_S)_i\right),$$

where $v_S := \bigwedge_{(I \leq v)} S$ and $w_T := \bigwedge_{(I \leq w)} T$, and with differentials as defined in 3.7. We obtain the relative Betti diagrams of M from the homology of this chain complex. For example, the relative projective resolution of M_0 is



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Paper B

Algebraic Wasserstein distances and stable homological invariants of data

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Algebraic Wasserstein distances and stable homological invariants of data

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Abstract

Distances have a ubiquitous role in persistent homology, from the direct comparison of homological representations of data to the definition and optimization of invariants. In this article we introduce a family of parametrized pseudometrics between persistence modules based on the algebraic Wasserstein distance defined by Skraba and Turner, and phrase them in the formalism of noise systems. This is achieved by comparing p-norms of cokernels (resp. kernels) of monomorphisms (resp. epimorphisms) between persistence modules and corresponding bar-to-bar morphisms, a novel notion that allows us to bridge between algebraic and combinatorial aspects of persistence modules. We use algebraic Wasserstein distances to define invariants, called Wasserstein stable ranks, which are 1-Lipschitz stable with respect to such pseudometrics. We prove a low-rank approximation result for persistence modules which allows us to efficiently compute Wasserstein stable ranks, and we propose an efficient algorithm to compute the interleaving distance between them. Importantly, Wasserstein stable ranks depend on interpretable parameters which can be learnt in a machine learning context. Experimental results illustrate the use of Wasserstein stable ranks on real and artificial data and highlight how such pseudometrics could be useful in data analysis tasks.

Keywords Persistent homology · Persistence modules · Stable topological invariants of data · Wasserstein metrics

Mathematics Subject Classification 55N31 · 62R40 · 55U99

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1 Introduction

While Topological Data Analysis (TDA) has historically focused on studying the global shape of data, persistent homology has since grown to provide popular techniques for incorporating both global topological features and local geometry into data analysis pipelines (Adams and Moy 2021). Through the lens of persistent homology, global topological features can be encoded by long bars in a barcode decomposition of the persistence module, while local geometric features are characterized by short bars in the barcode. Indeed, both the information of long bars and short bars in the barcode (Bendich et al. 2016; Hiraoka et al. 2016), as well as their location along the filtration scale (Stolz et al. 2017; Chachólski and Riihimäki 2020; Agerberg et al. 2021), turn out to be relevant in data analysis tasks. Introduced to persistent homology in Cohen-Steiner et al. (2010), Wasserstein distances offer a way to determine a trade-off between global and local features in persistence. Such distances have been widely used in applications and have been studied both from a combinatorial perspective and more recently with an algebraic approach (Bubenik et al. 2023; Skraba and Turner 2020). Wasserstein distances are parametrized by two parameters in $[1, \infty]$, commonly fixed to the values of $1, 2$, and ∞. One of the aims of this article is to define a richer family of parametrized Wasserstein distances where, in addition to standard parameters determining sensitivity to short bars globally in the parameter space, a *contour* is introduced to locally weight different parts of the parameter space. We propose that the optimal parameter values for a particular task should be learned in a machine learning context. Our contribution is part of more general efforts of identifying parametrized families of metrics and invariants for persistence (Bubenik et al. 2015; Scolamiero et al. 2017; Hofer et al. 2017; Zhao and Wang 2019; Carrière et al. 2020).

Algebraic Wasserstein distances.
The study of algebraic distances between persistence modules is an active research direction in TDA, as demonstrated by the recent works on amplitudes (Giunti et al. 2024) and exact weights (Bubenik et al. 2023). In this article we provide a new proof that the p-norm of a persistence module, introduced in Skraba and Turner (2020), defines a pseudometric for all $p \in [1, \infty]$. While (Skraba and Turner 2020) constructs a correspondence between the pseudometric induced by the p-norm and the Wasserstein distance between persistence diagrams, our proof shows that the p-norm determines a noise system (Scolamiero et al. 2017) and therefore an induced pseudometric. Our approach easily generalizes to define new pseudometrics on persistence modules, as for example the pseudometrics $d^q_{\mathcal{S}p,C}$ that combine p-norms with contours, effectively used as a reparametrization of the parameter space $[0, \infty)$. From a technical perspective, our framework requires to prove the axioms of noise systems without assuming that the p-norm induces a pseudometric (including the triangular inequality property) but rather studying how the p-norm interacts with monomorphisms, epimorphism, and short exact sequences. Among the axioms of noise systems, the one on short exact sequences (Lemma 4.15) is difficult to prove with our assumptions and to this purpose we introduce bar-to-bar morphisms, explained below.

It is interesting to see that Wasserstein distances fit in the noise system framework, as they are fundamentally different from noise systems that have been studied from a computational perspective so far. In fact, algorithms for the computation of stable ranks (that can be seen as vectorizations of persistence modules depending on the noise system) were only developed for so called *simple noise systems* (Chachólski and Riihimäki 2020; Gäfvert and Chachólski 2017). These noise systems have the extra property of being closed under direct sums, and can intuitively be thought of as being sensitive only to the longest bars, which leads to L^∞-type distances. The noise systems associated with algebraic Wasserstein distances for $p < \infty$ are of a different nature, and in particular they are not closed under direct sums.

From a practical and computational perspective, combinatorial distances between persistence diagrams are more straightforward to compute than algebraic distances between persistence modules. The combinatorial (p, C)-Wasserstein distances associated to $d^p_{\mathcal{S}^{p,C}}$ in Sect. 4.4 offer a convenient way to compute contour distances and the combination of contour and Wasserstein distances between persistence modules, relying on the already developed computational machinery for Bottleneck $(p = \infty)$ and Wasserstein distances between persistence diagrams. In this article, however, our focus is not on the computation of the Wasserstein distance between two given persistence modules, but on invariants called Wasserstein stable ranks defined and computed using the distances.

Bar-to-bar morphisms.

The approach carried out in this article for proving that p-norms of persistence modules satisfy the axioms of noise systems relies on comparing monomorphisms (resp. epimorphisms) between persistence modules and so-called *bar-to-bar* monomorphisms (resp. epimorphisms) between the same persistence modules. Intuitively, in a bar-to-bar morphism (see Definition 3.1) every bar in the barcode decomposition of the domain maps non-trivially to at most one bar in the barcode decomposition of the codomain. Bar-to-bar morphisms are thus much simpler than general morphisms of persistence modules, and we show that they can be used to effectively reduce algebraic problems to easier problems of combinatorial nature. In particular, an important problem related to the definition and construction of algebraic distances is the minimization of kernels and cokernels of morphisms (see e.g. Definition 4.3) with respect to a chosen notion of "size", which in this article is the p-norm of persistence modules (Sect. 2.6) or a more general notion of norm combining p-norms and contours (Definition 4.7). Our main theoretical results Theorem 3.14 and Theorem 3.16 state that for any monomorphism (resp. epimorphism) between two persistence modules there exists a bar-to-bar monomorphism (resp. epimorphism) between the same persistence modules whose cokernel (resp. kernel) has smaller or equal norm.

Various types of bar-to-bar morphisms can be constructed. For example, as we observe in Sect. 3.4, there are bar-to-bar monomorphisms and epimorphisms associated with the induced matchings of Bauer and Lesnick (2015). Bar-to-bar morphisms are however more general then the induced matchings, and are also fundamentally different from other notions of matchings, such as the sub-barcode matchings of Chubet et al. (2022). However, our bar-to-bar morphisms can be used as a tool to prove that the monomorphism (resp. epimorphism) associated to the induced matching has the cokernel (resp. kernel) with minimal p-norm among all monomorphisms (resp. epimorphisms) with the same domain and codomain. (Corollaries 3.20 and 3.22).

Several other key results of this article are proven leveraging Theorems 3.14 and 3.16. For example, we use these theorems to show that p-norms of persistence modules satisfy the axiom of noise systems on short exact sequences (Lemma 4.15). We also use our main results of Sect. 3 to prove a low-rank approximation result for persistence modules, analogous for example to the Eckart-Young-Mirsky theorem in the context of matrices (compare e.g. with Golub et al. 1987, Sect. 1 and references therein), where the notion of rank we use for persistence modules is the number of bars in the barcode decomposition. Given a persistence module X of rank k, for every $r \leq k$ we identify a persistence module of rank r that is closest to X in algebraic Wasserstein distance, and we express its distance from X (Propositions 4.30 and 4.32). These results and their generalization (Proposition 4.32) allow us to compute Wasserstein stable ranks (see Proposition 5.1), the invariants that we introduce in this article.

Wasserstein stable ranks, a class of learnable vectorizations.
The computation of Wasserstein distances between persistence modules remains expensive despite recent progress (Kerber et al. 2017), and the space of persistent modules is not directly amenable to statistical methods and machine learning. For these reasons, feature maps from persistence modules or diagrams have become an important component of the TDA machine learning pipeline. These techniques introduce a map between the space of persistence modules and a vector space where statistical and machine learning methods are well-developed. We propose a new class of feature maps, directly related to the Wasserstein distances $d^q_{\mathcal{S}p,C}$ between persistence modules, and with interpretable, learnable parameters. Having fixed a pseudometric in the family of Wasserstein distances $d^q_{\mathcal{S}p,C}$, the Wasserstein stable rank of a persistence module with respect to the chosen pseudometric can be explicitly computed with a formula (Proposition 5.1) derived from our results on monomorphisms and epimorphisms. The computational complexity of determining the Wasserstein stable rank is $O(n \log n)$ in the number n of bars of a persistence module.

A parametrized family of stable ranks can be obtained by varying the Wasserstein distances, opening up for the possibility to tune parameters for a particular task, resulting in feature maps that focus on the discriminative aspects of the persistence modules in a dataset. Previous learnable feature maps (Hofer et al. 2017; Carrière et al. 2020; Reinauer et al. 2021) make the choice of *expressiveness* (being able to learn any arbitrary function on the space of persistence modules) over *stability* (learning a function under the constraint that it is robust to perturbations of the input). Moreover, since the methods are often parametrized by complex neural networks, it is difficult to compare and interpret parametrizations learned for different tasks. Our Wasserstein stable ranks are stable by construction. More precisely, the interleaving distance between Wasserstein stable ranks is 1-Lipschitz with respect to the corresponding Wasserstein distance used in its construction. Similarly to Wasserstein stable ranks, we also provide a simple formula for computing the interleaving distance between them at the cost of $O(n \log n)$ in the maximum number of bars in the two persistence modules we are comparing.

We use a metric learning framework to learn an optimal parametrization for a problem at hand, observe that a better model can be obtained by jointly optimizing the parameters p and the ones related to the contour C and illustrate that the output can be readily interpreted in terms of the learned parametrization focusing on e.g. global/local features or various parts of the filtration scale. The methods are demonstrated on a synthetic and a real-world datasets.

Outline of the paper.
Section 2 contains background material. In Sect. 3 we prove results on the p-norm of the cokernel of a monomorphism and, dually, of the kernel of an epimorphism of persistence modules. Section 4 is a study of Wasserstein distances and their generalizations involving contours in the framework of noise systems. In Sect. 5 we compute Wasserstein stable ranks and interleaving distances between them, which we use to formulate a metric learning problem. In Sect. 6 we illustrate the use of Wasserstein stable ranks on synthetic and real-world data, learning optimal parameters of algebraic Wasserstein distances.

2 Preliminaries

2.1 Persistence modules and persistent homology

Let $[0, \infty)$ denote the totally ordered set of nonnegative real numbers, regarded as the category induced by the order structure. We consider an arbitrary fixed field K and denote by vect_K the category of finite dimensional vector spaces over K. A **persistence module over** K is a functor $X: [0, \infty) \to \mathrm{vect}_K$. Explicitly, X consists of a collection of finite dimensional vector spaces X_t for all t in $[0, \infty)$, together with a collection of linear functions $X_{s\leq t}: X_s \to X_t$, called **transition functions**, for all $s \leq t$ in $[0, \infty)$, such that $X_{s\leq t} X_{r\leq s} = X_{r\leq t}$ for all $r \leq s \leq t$, and $X_{t\leq t}$ is the identity function on X_t for all t in $[0, \infty)$. A **morphism** or natural transformation $f: X \to Y$ between two persistence modules X and Y is a collection of linear functions $f_t: X_t \to Y_t$, for all t in $[0, \infty)$, such that $f_t X_{s\leq t} = Y_{s\leq t} f_s$ for all $s \leq t$ in $[0, \infty)$.

A persistence module X is **tame** if there exist real numbers $0 = t_0 < t_1 < \cdots < t_k$ such that the transition function $X_{s\leq t}$ is a non-isomorphism only if $s < t_i \leq t$ for some $i \in \{1, \ldots, k\}$. We denote by Tame the category of tame persistence modules and morphisms between them. The class of objects of this category will be denoted by Tame as well.

Convention 2.1 In this article we always work in the category of tame persistence modules over a fixed field K. For brevity the term persistence module will be used to refer to tame persistence modules over K.

A morphism $f: X \to Y$ in Tame is a monomorphism (respectively, an epimorphism or isomorphism) if the linear functions $f_t: X_t \to Y_t$ are monomorphisms (respectively, epimorphisms or isomorphisms) of vector spaces, for all t in $[0, \infty)$. Kernels, cokernels and direct sums in Tame are defined componentwise. For example, for any persistence modules X and Y, the direct sum $X \oplus Y$ is the persistence module defined by $(X \oplus Y)_t = X_t \oplus Y_t$ and $(X \oplus Y)_{s \leq t} = X_{s \leq t} \oplus Y_{s \leq t}$, for all $s \leq t$ in $[0, \infty)$. The zero persistence module or **zero module**, i.e., the functor identically equal to the zero vector space on objects, will be denoted by 0.

Let $a < b$ in $[0, \infty]$. We denote by $K(a, b)$ the persistence module defined as follows: for any t in $[0, \infty)$,

$$K(a, b)_t := \begin{cases} K & \text{if } a \leq t < b \\ 0 & \text{otherwise,} \end{cases}$$

and for any $s \leq t$ in $[0, \infty)$,

$$K(a, b)_{s \leq t} := \begin{cases} \mathrm{id}_K & \text{if } K(a, b)_s = K = K(a, b)_t \\ 0 & \text{otherwise.} \end{cases}$$

We call $K(a, b)$ the **bar** (or interval module) with **start-point** a and **end-point** b. We say that the bar $K(a, b)$ is **infinite** if $b = \infty$ and **finite** otherwise. We say that the left-closed, right-open interval $[a, b)$ in $[0, \infty)$ is the **support** of the bar $K(a, b)$. As an easy consequence of naturality, a morphism $f : K(a_1, b_1) \to K(a_2, b_2)$ between bars can be nonzero (i.e. have some component f_a different from the zero map) only if $a_2 \leq a_1 < b_2 \leq b_1$. In this case, $\ker f$ is isomorphic to $K(b_2, b_1)$ if $b_2 < b_1$, and is zero otherwise, and coker f is isomorphic to $K(a_2, a_1)$ if $a_2 < a_1$, and is zero otherwise.

A persistence module is **indecomposable** if, whenever it is isomorphic to a direct sum $Y \oplus Z$ with Y and Z in Tame, either $Y = 0$ or $Z = 0$. Bars are indecomposable and, as the following fundamental result implies, any indecomposable in Tame is isomorphic to a bar. We refer the reader to Chazal et al. (2016) for more details on the algebraic structure of persistence modules.

Theorem 2.2 (Structure of persistence modules) *Any (tame) persistence module X is isomorphic to a finite direct sum of bars of the form $\bigoplus_{i=1}^{k} K(a_i, b_i)$, with $a_i < b_i$ in $[0, \infty]$ for every $i \in \{1, \dots, k\}$. This decomposition is unique up to permutation: if $X \cong \bigoplus_{i=1}^{k} K(a_i, b_i) \cong \bigoplus_{j=1}^{\ell} K(c_j, d_j)$, then $k = \ell$ and there exists a permutation σ on $\{1, \dots, k\}$ such that $a_i = c_{\sigma(i)}$ and $b_i = d_{\sigma(i)}$, for every $i \in \{1, \dots, k\}$.*

A decomposition of a persistence module X as a direct sum of bars as in Theorem 2.2 is called a **barcode decomposition** of X. In this article, we will occasionally denote a barcode decomposition of X by $\bigoplus_{i=1}^{k} X_i$ when we do not need an explicit notation for the bars' start- and end-points. The number k of bars in any barcode decomposition of X is called the **rank** of X, denoted by $\mathrm{rank}(X)$.

Given a persistence module X, consider an element $x \in X_a$ for some a in $[0, \infty)$, and let $b := \sup\{t \in [a, \infty) \mid X_{a \leq t}(x) \neq 0\}$ in $[a, \infty]$. The element x is called a **generator** of X if the morphism $g : K(a, b) \to X$ defined by $g_a(1) = x$ is such that the composition rg with some morphism $r : X \to K(a, b)$ is the identity on $K(a, b)$. We call $K(a, b)$ the **bar generated by** x, and we observe that it is a direct summand of X. We call a collection of elements $\{x_i \in X_{a_i}\}_{i=1}^{k}$ a **set of generators** of X if each

x_i generates a bar $K(a_i, b_i)$ and the morphisms $g_i : K(a_i, b_i) \to X$ defined by x_i induce an isomorphism $\bigoplus_{i=1}^{k} K(a_i, b_i) \to X$.

As we will use basic homological algebra methods in Tame, we remark that infinite bars $K(a, \infty)$, for all a in $[0, \infty)$, are free in Tame, and that the notions of free and projective coincide in Tame (see Bubenik and Milićević 2021 for details). Any bar $K(a, b)$ with $b < \infty$ admits a minimal free resolution of the form $0 \to K(b, \infty) \to K(a, \infty) \to K(a, b) \to 0$.

Remark 2.3 We note that $\mathrm{rank}(X)$ can be viewed as a classical homological invariant corresponding to the number of generators in a minimal free resolution of X, which yields an alternative definition of the rank that is applicable to multiparameter persistence modules (Scolamiero et al. 2017).

Lastly, let us briefly comment on a set theoretical detail regarding the category Tame. In Tame, the class of isomorphism classes of objects is a set, as a consequence of Theorem 2.2. In this article, we consider some class functions defined on Tame, and we occasionally refer to them simply as functions for brevity. Since all class functions on Tame we consider are constant on isomorphism classes of objects, they can be regarded as proper functions defined on the set of isomorphism classes of persistence modules.

2.2 Contours

Contours can be thought of as describing coherent ways to "flow" across the parameter space $[0, \infty)$ of persistence modules. In this article, we call **contour** a function $C : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that, for all a, b, ε, τ in $[0, \infty)$, the following inequalities hold:

1. if $a \le b$ and $\varepsilon \le \tau$, then $C(a, \varepsilon) \le C(b, \tau)$;
2. $a \le C(a, 0)$;
3. $C(C(a, \varepsilon), \tau) \le C(a, \varepsilon + \tau)$.

In Gäfvert and Chachólski (2017) contours are defined in the case of n-parameter persistence modules. Contours are further studied for 1-parameter persistence in Chachólski and Riihimäki (2020), where several concrete examples are given. In Chachólski and Riihimäki (2020), the definition of contour is slightly more general than ours; for example, $C(a, \varepsilon)$ can take the value ∞. Similar notions to contours appear in the literature by the name of *superlinear families of tranlations* (Bubenik et al. 2015) and *flows on posets* (de Silva et al. 2018).

A contour C is called an **action** if the inequalities of (2.) and (3.) are equalities, that is, if $a = C(a, 0)$ and $C(C(a, \varepsilon), \tau) = C(a, \varepsilon + \tau)$, for all a, ε, τ. A contour C is **regular** (Chachólski and Riihimäki 2020) if the following conditions hold:

- $C(-, \varepsilon) : [0, \infty) \to [0, \infty)$ is a monomorphism for all $\varepsilon \in [0, \infty)$;
- $C(a, -) : [0, \infty) \to [0, \infty)$ is a monomorphism whose image is $[a, \infty)$, for all $a \in [0, \infty)$.

The second condition of regular contours ensures that $C(a, 0) = a$, for any a in $[0, \infty)$, and that C is strictly increasing in the second variable: $C(a, \varepsilon) < C(a, \tau)$

whenever $\varepsilon < \tau$, for any a in $[0, \infty)$. For brevity, we call a contour C a **regular action** if it is both regular and an action.

Let C be a regular contour. For all $a \in [0, \infty)$, we define the function $\ell(a, -)$ to be the inverse of the function $C(a, -)\colon [0, \infty) \to [a, \infty)$, that is, $\ell(a, b) = C(a, -)^{-1}(b)$ for any $b \in [a, \infty)$, and we set $\ell(a, \infty) = \infty$. We call ℓ the **lifetime function** associated with C. We observe that, since regular contours are injective functions in the second variable, $\ell(a, b)$ is well-defined for every pair $a \leq b$. Throughout the article, the **lifetime of a bar $K(a, b)$ with respect to a contour** C is the value $\ell(a, b)$ of the lifetime function associated with C.

As a first example of contour we consider the **standard contour**, i.e. the function D defined by $D(a, \varepsilon) = a + \varepsilon$, for every $a, \varepsilon \in [0, \infty)$. Informally, the standard contour describes the most uniform way to flow in the parameter space $[0, \infty)$ of a persistence module, linearly with unitary speed. We now introduce a large family of contours, called **integral contours of distance type** (Chachólski and Riihimäki 2020; Agerberg et al. 2021), parametrized by certain real-valued functions. Let $f\colon [0, \infty) \to (0, \infty)$ be a Lebesgue measurable function, called here a **density**. For every $a, \varepsilon \in [0, \infty)$, let $D_f(a, \varepsilon)$ be the real number in $[a, \infty)$ such that

$$\varepsilon = \int_a^{D_f(a,\varepsilon)} f(x)\, dx,$$

which is uniquely defined since f takes strictly positive values. The function $D_f\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ is a contour; moreover, it is regular and an action. We observe that, if the density f is the constant function with value 1, the distance type contour D_1 coincides with the standard contour.

2.3 Noise systems

Noise systems provide a way to quantify the size of persistence modules and to produce pseudometrics on Tame by comparing their sizes (Scolamiero et al. 2017). A **noise system** on Tame is a sequence $\mathcal{S} = \{\mathcal{S}_\varepsilon\}_{\varepsilon \in [0,\infty)}$ of subclasses of Tame such that:

- $0 \in \mathcal{S}_\varepsilon$, for all ε,
- $\mathcal{S}_\tau \subseteq \mathcal{S}_\varepsilon$ whenever $\tau \leq \varepsilon$,
- if $0 \to X_0 \to X_1 \to X_2 \to 0$ is a short exact sequence in Tame, then:
  - if $X_1 \in \mathcal{S}_\varepsilon$, then $X_0, X_2 \in \mathcal{S}_\varepsilon$,
  - if $X_0 \in \mathcal{S}_\varepsilon$ and $X_2 \in \mathcal{S}_\tau$, then $X_1 \in \mathcal{S}_{\varepsilon+\tau}$.

Given a noise system $\mathcal{S} = \{\mathcal{S}_\varepsilon\}_{\varepsilon \in [0,\infty)}$ it is natural to associate to each persistence module X the smallest ε such that $X \in \mathcal{S}_\varepsilon$. This defines a class function $\alpha_{\mathcal{S}}\colon \mathsf{Tame} \to [0, \infty]$ called in Giunti et al. (2024) the **amplitude** associated to $\mathcal{S}$.

A noise system $\mathcal{S} = \{\mathcal{S}_\varepsilon\}_{\varepsilon \in [0,\infty)}$ is **closed under direct sums** if $X \oplus Y \in \mathcal{S}_\varepsilon$ whenever $X, Y \in \mathcal{S}_\varepsilon$, for every $\varepsilon \in [0, \infty)$. Contours (Sect. 2.2) provide examples of noise systems satisfying this property. Given a contour C and any $\varepsilon \in [0, \infty)$, let

$$\mathcal{S}_\varepsilon := \{X \in \mathsf{Tame} \mid X_{a \leq C(a,\varepsilon)} = 0 \text{ for all } a \in [0, \infty)\}.$$

It is proved in (Gäfvert and Chachólski 2017, Prop. 9.4) that the sequence $\{\mathcal{S}_\varepsilon\}_{\varepsilon\in[0,\infty)}$ defined in this way is a noise system closed under direct sums. In particular, the noise system induced by the standard contour has components

$$\mathcal{S}_\varepsilon := \{X \in \mathsf{Tame} \mid X_{a\leq a+\varepsilon} = 0 \text{ for all } a \in [0,\infty)\},$$

and coincides with the **standard noise system** introduced in Scolamiero et al. (2017).

2.4 Pseudometrics between persistence modules

In this article, we call (extended) **pseudometric** on Tame a class function d assigning to any pair of persistence modules X, Y in Tame an element $d(X, Y) \in [0,\infty]$ such that the following conditions hold for any X, Y, Z:

- $d(X, Y) = d(Y, X)$,
- $d(X, Y) = 0$ whenever X is isomorphic to Y,
- $d(X, Z) \leq d(X, Y) + d(Y, Z)$.

The third condition, known as the triangle inequality, combined with the second one yields $d(X, Y) = d(X', Y')$ whenever $X \cong X'$ and $Y \cong Y'$. This definition of pseudometric coincides with Definition 3.3 in Bubenik et al. (2023) when considering the category Tame.

We now briefly explain how noise systems yield pseudometrics on Tame. Let $\mathcal{S}$ be a noise system on Tame. For any $\varepsilon \in [0,\infty)$, we say that two persistence modules X and Y are ε-**close** if there exists a persistence module Z and a pair of morphisms $X \xleftarrow{f} Z \xrightarrow{g} Y$ such that

$$\ker f \in \mathcal{S}_{\varepsilon_1}, \quad \operatorname{coker} f \in \mathcal{S}_{\varepsilon_2}, \quad \ker g \in \mathcal{S}_{\varepsilon_3}, \quad \operatorname{coker} g \in \mathcal{S}_{\varepsilon_4},$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,\infty)$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \leq \varepsilon$. Define

$$d_{\mathcal{S}}(X, Y) := \inf\{\varepsilon \in [0,\infty) \mid X \text{ and } Y \text{ are } \varepsilon\text{-close}\},$$

adopting the convention $\inf \varnothing = \infty$. As shown in (Scolamiero et al. 2017, Prop. 8.7), $d_{\mathcal{S}}$ is a pseudometric on Tame.

We remark that the pseudometric $d_{\mathcal{S}}$ associated with the standard noise system is equivalent to the interleaving distance (Lesnick 2015), as proved by (Gäfvert and Chachólski (2017), Prop. 12.2).

2.5 Hierarchical stabilization and stable rank

In the context of TDA, **hierarchical stabilization** is a method to convert a discrete invariant of persistence modules into a stable invariant suitable for data analysis. This technique has been studied in Scolamiero et al. (2017), Gäfvert and Chachólski (2017) in the case of multiparameter persistence modules, and has been further investigated in Chachólski and Riihimäki (2020) in the case of one-parameter persistence. Hierarchical stabilization has a very general formulation, which allows for several choices of discrete invariants, and in principle is not restricted to categories of persistence modules. For the hierarchical stabilization of the rank, also called stable rank, some computational methods have been developed (Chachólski and Riihimäki 2020; Gäfvert and Chachólski 2017). In this article we will restrict our attention to the stable rank and further develop its computation.

Besides choosing a discrete invariant, hierarchical stabilization requires the choice of a pseudometric between persistence modules, which plays an active role in calculating the corresponding stable invariant. Consider the rank of a persistence module (Sect. 2.1) as a class function $\text{rank}\colon \mathsf{Tame} \to \mathbb{N}$ mapping any persistence module X to the natural number $\text{rank}(X)$.

Definition 2.4 Given a pseudometric d on Tame (Sect. 2.4), the **stable rank** of a persistence module X with respect to the pseudometric d is the function $\widehat{\text{rank}}_d(X)\colon [0,\infty) \to [0,\infty)$ defined, for all $t \in [0,\infty)$, by

$$\widehat{\text{rank}}_d(X)(t) := \min\{\text{rank}(Y) \mid Y \in \mathsf{Tame} \text{ and } d(X,Y) \le t\}.$$

We observe that the function $\widehat{\text{rank}}_d(X)$ is non-increasing and takes values in $\mathbb{N}$, so it belongs to the set $\mathcal{M}$ of Lebesgue measurable functions $[0,\infty) \to [0,\infty)$.

To illustrate the stability of the invariant $\widehat{\text{rank}}_d$, we consider a pseudometric $d_{\bowtie}$ on $\mathcal{M}$, called the **interleaving distance**, defined for all $f, g \in \mathcal{M}$ by

$$d_{\bowtie}(f,g) := \inf\{\varepsilon \in [0,\infty) \mid f(t) \ge g(t+\varepsilon) \text{ and } g(t) \ge f(t+\varepsilon),\ \text{ for all } t \in [0,\infty)\},$$

setting by convention $\inf \varnothing = \infty$. The stable rank then satisfies the following Lipschitz condition.

Proposition 2.5 (Scolamiero et al. 2017) *Let d be a pseudometric on* Tame*, and let X, Y be persistence modules. Then $d(X,Y) \ge d_{\bowtie}(\widehat{\text{rank}}_d(X), \widehat{\text{rank}}_d(Y))$.*

2.6 *p*-norms

In this subsection, we briefly review properties of p-norms that are useful for our work. For $p \in [1,\infty]$, the p-norm (also called L^p-norm) on $\mathbb{R}^n$ is the function $\|\cdot\|_p : \mathbb{R}^n \to [0,\infty)$ defined, for each $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, by

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} & \text{for } p \in [1,\infty) \\ \max\{|x_i|\}_{i\in\{1,\ldots,n\}} & \text{for } p = \infty. \end{cases}$$

We note that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$, for all $x \in \mathbb{R}^n$. The triangle inequality (or subadditivity condition) $\|x+y\|_p \le \|x\|_p + \|y\|_p$, for all $x, y \in \mathbb{R}^n$, is also referred to as Minkowski inequality.

A fundamental property of p-norms on $\mathbb{R}^n$ is the following: for $x \in \mathbb{R}^n$ and for $1 \leq p \leq q \leq \infty$, the inequalities

$$\|x\|_q \leq \|x\|_p \leq n^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|x\|_q \tag{1}$$

hold and are sharp, where by convention we set $\frac{1}{\infty} = 0$. We refer to the first inequality as the monotonicity property of p-norms.

The following elementary property of p-norms is useful in this work: for $p \in [1, \infty]$, if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ and $z = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$, then

$$\left\| \left(\|x\|_p \, , \, \|y\|_p \right) \right\|_p = \|z\|_p \, . \tag{2}$$

Finally, let us also observe that p-norms are permutation invariant, and that they preserve the order on $[0, \infty)^n$, meaning that if $x \leq y$ in $[0, \infty)^n$ according to the coordinate-wise order, then $\|x\|_p \leq \|y\|_p$.

In this article, we generally consider p-norms as functions from $[0, \infty]^n$ to $[0, \infty]$, extending the usual definition by setting $\|x\|_p = \infty$ whenever x has some coordinate $x_i = \infty$. All properties stated above still hold with this definition.

Following (Skraba and Turner 2020), we will consider p-norms of persistence modules, whose definition relies on the barcode decomposition (Sect. 2.1). Such norms are already introduced in (Chen and Edelsbrunner 2011) for persistence diagrams. For $p \in [1, \infty]$, the p-norm of a persistence module X having barcode decomposition $X \cong \bigoplus_{i=1}^k K(a_i, b_i)$ is defined by

$$\|X\|_p := \begin{cases} \left(\sum_{i=1}^k |b_i - a_i|^p \right)^{\frac{1}{p}} & \text{for } p \in [1, \infty) \\ \max\{|b_i - a_i|\}_{i \in \{1, \ldots, k\}} & \text{for } p = \infty. \end{cases}$$

For $p \in [1, \infty]$ and $\varepsilon \in [0, \infty)$, the class of tame persistence modules with p-norm smaller or equal to ε is denoted by:

$$\mathcal{S}^p_\varepsilon := \{X \in \mathsf{Tame} \mid \|X\|_p \leq \varepsilon\},$$

and we set $\mathcal{S}^p := \{\mathcal{S}^p_\varepsilon\}_{\varepsilon \in [0, \infty)}$.

3 Monomorphisms, epimorphisms, and their *p*-norms

In this section we introduce bar-to-bar morphisms between persistence modules (Definition 3.1), which can informally be described as morphisms such that every bar in the barcode decomposition of the domain maps non-trivially to at most one bar in the barcode decomposition of the codomain. Our aim is proving results (Theorems 3.14 and 3.16) which compare monomorphisms and epimorphisms between two persistence modules to bar-to-bar monomorphisms and epimorphisms between the same persistence modules. These results allow us to reduce algebraic problems to much simpler combinatorial problems, as shown for example in Corollaries 3.20 and 3.22.

3.1 Free presentations of monomorphisms

Given a monomorphism $f: Z \hookrightarrow X$ between persistence modules, we want to determine the barcode decomposition of coker f. We briefly describe a method that uses free resolutions of the persistence modules Z and X.

Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & R_Z \arrow[r, "i_Z"] \arrow[d, "f_R"] & G_Z \arrow[r, "p_Z"] \arrow[d, "f_G"] & Z \arrow[r] \arrow[d, hook, "f"] & 0 \\
0 \arrow[r] & R_X \arrow[r, "i_X"] & G_X \arrow[r, "p_X"] & X \arrow[r] \arrow[d, two heads, "q"] & 0 \\
& & & \operatorname{coker} f &
\end{tikzcd}$$

where the rows are (minimal) free resolutions of the persistence modules Z and X respectively, and q denotes the canonical epimorphism. The given morphism f induces a morphism $f_G: G_Z \to G_X$ between the modules of generators and a morphism $f_R: R_Z \to R_X$ between the modules of relations that make the diagram commutative (see e.g. Rotman 2009, Thm. 6.16). We have coker $f \cong \operatorname{coker}([f_G\ i_X]: G_Z \oplus R_X \to G_X)$, where the morphism $[f_G\ i_X]$ sends $(z, r) \in G_Z \oplus R_X$ to $f_G(z) + i_X(r)$. The isomorphism of cokernels is easy to prove, for example observing that the image of the composition qp_X is coker f given that both q and p_X are surjective and verifying via diagram chasing that its kernel coincides with the image of $[f_G\ i_X]: G_Z \oplus R_X \to G_X$.

In other words, we have a free presentation of coker f

$$G_Z \oplus R_X \xrightarrow{[f_G\ i_X]} G_X \twoheadrightarrow \operatorname{coker} f,$$

and we can use it to determine the barcode decomposition of coker f. More precisely, observing that coker f is isomorphic to the homology at the middle term of the free chain complex

$$G_Z \oplus R_X \xrightarrow{[f_G\ i_X]} G_X \longrightarrow 0,$$

we can compute the barcode decomposition of coker f by using the persistent homology algorithm on a matrix M_f representing the morphism $[f_G\ i_X]$, as we detail in Sect. 3.2. The persistent homology algorithm determines "pairings" of the basis elements of $G_Z \oplus R_X$ with the basis elements of G_X, which corresponds to the start- and end-point pairs of the bars of coker f.

In this section, we are interested in particular morphisms between persistence modules, which we call bar-to-bar morphisms.

Definition 3.1 A morphism $f\colon Z \to X$ of persistence modules is **bar-to-bar** if there are barcode decompositions $Z = \bigoplus_{i=1}^{m} Z_i$ and $X = \bigoplus_{j=1}^{n} X_j$ and there exist a subset $I \subseteq \{1, \ldots, m\}$ and an injective function $\alpha\colon I \to \{1, \ldots, n\}$ such that

$$f = \bigoplus_{i \in I} f_i \oplus \bigoplus_{i \in \{1,\ldots,m\} \setminus I} g_i \oplus \bigoplus_{j \in \{1,\ldots,n\} \setminus \alpha(I)} h_j, \tag{3}$$

where each $f_i := f|_{Z_i}$ is a nonzero morphism $Z_i \to X_{\alpha(i)}$, and where g_i denotes the zero morphism $Z_i \to 0$ and h_j denotes the zero morphism $0 \to X_j$.

Remark 3.2 If f is a bar-to-bar morphism as in (3), then $\ker f$ and $\operatorname{coker} f$ are easily determined recalling the case of a morphism between two bars (see Sect. 2.1), namely:

$$\ker f = \bigoplus_{i \in I} \ker f_i \oplus \bigoplus_{i \in \{1,\ldots,m\} \setminus I} Z_i, \qquad \operatorname{coker} f = \bigoplus_{i \in I} \operatorname{coker} f_i \oplus \bigoplus_{j \in \{1,\ldots,n\} \setminus \alpha(I)} X_j.$$

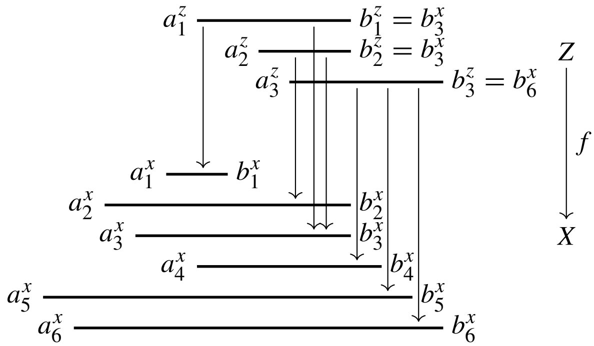
Furthermore, if f is a monomorphism, the fact that $\ker f$ vanishes implies that $I = \{1, \ldots, m\}$, and the existence of the injective function α implies $m \le n$. Dually, $\alpha(I) = \{1, \ldots, n\}$ and $n \le m$ if f is an epimorphism.

The main result of this section is the following (Theorem 3.14): given any monomorphism $f\colon Z \hookrightarrow X$, there is a bar-to-bar monomorphism $f_b\colon Z \hookrightarrow X$ such that $\|\operatorname{coker} f_b\|_p \le \|\operatorname{coker} f\|_p$ for any $p \in [1, \infty]$. A dual statement (Theorem 3.16) holds for kernels of epimorphisms.

3.2 Finding monomorphisms with smaller cokernels

To prove our inequalities between p-norms of cokernels, we modify a strategy used in (Skraba and Turner 2020, Sect. 7.1) to obtain new inequalities between p-norms of persistence modules, based on the rearrangement inequality (Theorem 3.12) and on the comparison of pairings in certain barcode decompositions using the persistent homology algorithm. For simplicity, we fix the field with two elements $\mathbb{F}_2$ as the base field in this subsection, but our results work for any base field.

Let Z and X be persistence modules and $f\colon Z \hookrightarrow X$ a monomorphism of persistence modules. Fix $\{z_i\}_{i=1}^{m}$ and $\{x_j\}_{j=1}^{n}$ sets of generators of Z and X, respectively, and denote by $Z = \bigoplus_{i=1}^{m} K(a_i^z, b_i^z)$ and $X = \bigoplus_{j=1}^{n} K(a_j^x, b_j^x)$ the respective barcode decompositions. That is, for every z_i, a_i^z is the **degree** of $z_i \in Z_{a_i^z}$ and b_i^z is the end-point of the bar generated by z_i, and similarly for the x_j. In this section, we assume for the ease of exposition that X has no infinite bars in its decomposition. All the results we present can be adapted to the general case by setting $b_j^x = \infty$ whenever x_j generates an infinite bar. Throughout this section, we will consider an example monomorphism f, which we represent as follows:



(4)

The persistence modules Z and X are represented in terms of their barcode decompositions. An arrow between bars indicates that the bar in the domain maps non-trivially to the bar in the codomain.

The main results of this subsection are based on matrix reduction arguments applied to a matrix M_f associated with the morphism $[f_G\ i_X]\colon G_Z \oplus R_X \to G_X$ (Sect. 3.1), which we construct as follows.

Definition 3.3 Define the sets of labels $\mathcal{G}_X := \{x_j\}_{j=1}^n$, $\mathcal{G}_Z := \{z_i\}_{i=1}^m$, and $\mathcal{R}_X := \{r_j\}_{j=1}^n$, where $\{z_i\}_{i=1}^m$ and $\{x_j\}_{j=1}^n$ are generators of Z and X respectively and r_j corresponds to the generator of R_X that is sent by i_X to the bar generated by x_j in G_X. The **degree** of r_j is b_j^x.

The **presentation matrix of** f is an $n \times (m+n)$ matrix M_f with rows labeled by $\mathcal{G}_X$ and columns labeled by $\mathcal{G}_Z \sqcup \mathcal{R}_X$, constructed as follows. For each z_i in $\mathcal{G}_Z$, we set the corresponding column of M_f to be the column vector $f_{a_i^z}(z_i) \in X_{a_i^z}$ in the basis given by the nonzero elements of $\{X_{a_j^x \le a_i^z}(x_j)\}_{j=1}^n$. Note that if $X_{a_j^x \le a_i^z}(x_j)$ is 0, then $M_f(x_j, z_i)$ is also 0. For each r_j in $\mathcal{R}_X$, we set the corresponding column of M_f to be the zero vector except with a 1 on the row x_j. Finally, we reorder the rows and columns so that the degrees of the labels are nondecreasing.

We denote by $M_f(x, c)$ the entry of M_f in row $x \in \mathcal{G}_X$ and column $c \in \mathcal{G}_Z \sqcup \mathcal{R}_X$.

As an example, one presentation matrix of the example monomorphism f from (4) is

$$M_f = \begin{array}{c} \\ x_5 \\ x_6 \\ x_2 \\ x_3 \\ x_1 \\ x_4 \end{array} \begin{array}{c} \begin{array}{ccccccccc} z_1 & r_1 & z_2 & z_3 & r_2 & r_3 & r_4 & r_5 & r_6 \end{array} \\ \left[\begin{array}{c|c|cc|ccccc} 0 & & 0 & 1 & & & & 1 & \\ 0 & & 0 & 1 & & & & & 1 \\ 0 & & 1 & 0 & 1 & & & & \\ 1 & & 1 & 0 & & 1 & & & \\ 1 & 1 & 0 & 0 & & & & & \\ 0 & & 0 & 1 & & & 1 & & \end{array}\right] \end{array}, \tag{5}$$

where the columns $\mathcal{G}_Z = \{z_1, z_2, z_3\}$ are outlined, while the columns $\mathcal{R}_X = \{r_1, \ldots, r_6\}$ are represented sparsely: blank spaces are zero coefficients. Note that the restriction of the matrix M_f to the columns $\mathcal{R}_X$ represents the morphism $i_X : R_X \to G_X$.

Remark 3.4 As we mentioned in Sect. 3.1, we want to determine the barcode decomposition of coker f by using the persistent homology algorithm on the matrix M_f representing the morphism $[f_G \ i_X]$. More precisely, we are interested in methods to compute barcode decompositions based on matrix reduction via left-to-right column operation, like the so-called standard algorithm for persistent homology (Edelsbrunner et al. 2000; Zomorodian and Carlsson 2005) (see Algorithm 1 in Otter et al. 2017 for a description). Even though these methods are usually presented for filtered simplicial complexes in the literature, they extend to graded free chain complexes as in our case. The barcode decomposition (of coker f in our case) can be read out from a reduced matrix, and does not depend on the way of reducing the matrix via left-to-right column operations (see Lemma 3.5).

Let $\bar{M}_f$ be a complete reduction of M_f by left-to-right column transformations, where a matrix is said to be **reduced** if no two columns have their lowest nonzero entry on the same row. Let σ_f be the function that to the k^{th} nonzero column of $\bar{M}_f$ associates the row of its lowest nonzero entry, for every $k \in \{1, \ldots, n\}$. We know that σ_f is a permutation on $\{1, \ldots, n\}$ since the n columns of M_f in $\mathcal{R}_X$ are linearly independent. In this section, we use square brackets for a permutation $\sigma = [\sigma(1) \cdots \sigma(n)]$ on $\{1, \ldots, n\}$ expressed in one-line notation, to distinguish it from the notation for cycles, denoted by $(c_1\, c_2\, \cdots\, c_\ell)$. For the running example (5), we get

$$\bar{M}_f = \begin{bmatrix} 0 & & 0 & 1 & & \boxed{1} \\ 0 & & 0 & 1 & \boxed{1} & 0 \\ 0 & & \boxed{1} & 0 & 0 & \\ 1 & \boxed{1} & 0 & 0 & 0 & \\ \boxed{1} & 0 & 0 & 0 & & \\ 0 & & 0 & \boxed{1} & 0 & \end{bmatrix},$$

where we have outlined the lowest nonzero coefficient of each column, and so $\sigma_f = [543621]$. We do not need to specify the order of transformations in this reduction thanks to the following lemma, which is a consequence of the pairing uniqueness lemma of (Cohen-Steiner et al. 2006, Sect. 3).

Lemma 3.5 *The permutation σ_f is well-defined. In particular, it does not depend on the choice of a sequence of left-to-right column operations to obtain a reduced matrix from M_f.*

By design of the persistent homology algorithm, a barcode decomposition of coker f is completely determined by σ_f together with the degrees of the generators of Z and X. In Corollary 3.11 we will provide a precise statement.

From the matrix M_f we define the **bar-to-bar matrix** M_b by Algorithm 1. The bar-to-bar matrix M_b is the presentation matrix of a **bar-to-bar monomorphism** $f_b : Z \hookrightarrow X$ having the same domain and codomain as f.

Algorithm 1 also partially reduces M_f and constructs an injective function $r_{\max}\colon \mathcal{G}_Z \to \mathcal{R}_X$. Given a column z in $\mathcal{G}_Z$, we call $r_{\max}(z)$ its **rightmost matched column** in $\mathcal{R}_X$. Informally, Algorithm 1 computes the bar-to-bar matrix M_b by setting to zero each entry in the columns z of M_f in $\mathcal{G}_Z$ except for the nonzero entry on the unique row x such that $M_f(x, r_{\max}(z)) = 1$. For example, starting with the matrix M_f from (5), we get

$$\left[\begin{array}{|c|c|cc|ccccc}
0 & & 0 & 1 & & & & 1 & \\
0 & & 0 & 1 & & & & & \boxed{1} \\
0 & & 1 & & \boxed{1} & & & & \\
1 & & & & & \boxed{1} & & & \\
1 & 1 & 0 & 0 & & & & & \\
0 & & 0 & 1 & & & 1 & &
\end{array}\right],$$

where the arrows represent the function $r_{\max}$. The corresponding matrix M_b is

$$\left[\begin{array}{|c|c|cc|ccccc}
0 & & 0 & 0 & & & & 1 & \\
0 & & 0 & 1 & & & & & 1 \\
0 & & 1 & 0 & 1 & & & & \\
1 & & 0 & 0 & & 1 & & & \\
0 & 1 & 0 & 0 & & & & & \\
0 & & 0 & 0 & & & 1 & &
\end{array}\right].$$

Algorithm 1 Bar-to-bar algorithm

Input: a presentation matrix M_f of a monomorphism f
Output: a partially reduced matrix M_f^*, the associated bar-to-bar matrix M_b, and a function $r_{\max}\colon \mathcal{G}_Z \to \mathcal{R}_X$

Let $M_b := M_f$
Let $M_f^* := M_f$
Set the columns $\mathcal{G}_Z$ of M_b to 0
for $r \in \mathcal{R}_X$ in decreasing order **do**
  Let x be the row associated to r (that is, $M_f^*(x, r) = 1$)
  if $\exists z \in \mathcal{G}_Z$ such that $M_f^*(x, z) = 1$ and $r_{\max}(z)$ is undefined **then**
    Let z be minimal such that $M_f^*(x, z) = 1$ and $r_{\max}(z)$ is undefined
    Set $M_b(x, z) = 1$
    Define $r_{\max}(z) := r$
    for $z' > z$ such that $M_f^*(x, z') = 1$ **do**
      Reduce column z' in M_f^* by column z to set to zero the entry in row x
      for $r' \in \mathcal{R}_X$ and x' the row associated to r', such that $r' < z'$ and $M_f^*(x', z') = 1$ **do**
        Reduce column z' in M_f^* by column r'
      end for
    end for
  end if
end for

The following two propositions prove useful facts regarding Algorithm 1.

Proposition 3.6 *In a presentation matrix M_f of a monomorphism $f : Z \hookrightarrow X$, all columns in $\mathcal{G}_Z$ are nonzero. Moreover, for every column $z \in \mathcal{G}_Z$, all columns in the set*

$$\Gamma(z) := \{r \in \mathcal{R}_X \mid r \text{ and } z \text{ have a nonzero entry on the same row}\}$$

have degree strictly larger than the degree of z, and $|\Gamma(z)|$ equals the number of nonzero entries of z.

Proof Since f is a monomorphism, it cannot send a generator of a bar of Z to zero, hence the columns in $\mathcal{G}_Z$ are nonzero. A nonzero entry in a column $z \in \mathcal{G}_Z$ indicates that the corresponding generator of a bar of Z maps non-trivially to the vector space generated by $X_{a^x \le a^z}(x)$ for some x generating a bar in X, where a^x is the degree of x and a^z is the degree of z. This implies that the end-point of the bar of X generated by x has degree strictly larger than the degree of z. Lastly, the cardinality of $\Gamma(z)$ equals the number of nonzero entries of z because the columns in $\mathcal{R}_X$ form a permutation matrix of rank n. □

Proposition 3.7 *Let $f : Z \hookrightarrow X$ be a monomorphism and let M_f be a presentation matrix of f. The execution of Algorithm 1 on M_f returns a well-defined function $r_{\max} : \mathcal{G}_Z \to \mathcal{R}_X$ that is injective. Furthermore, for every column $z \in \mathcal{G}_Z$, the column $r_{\max}(z)$ is to the right of z.*

Proof We prove that, for every column $z \in \mathcal{G}_Z$, $r_{\max}(z)$ is well-defined and to the right of z. We proceed by induction on a natural number $m \ge 0$, proving the result for all monomorphisms $f : Z \hookrightarrow X$ with presentations such that $|\mathcal{G}_Z| = m$.

If $m = 1$ and $\mathcal{G}_Z = \{z\}$, then the algorithm sets $r_{\max}(z)$ to be the rightmost column in $\mathcal{R}_X$ having a nonzero entry on the same row as a nonzero entry of z, which exists and is to the right of z by Proposition 3.6.

Now suppose that the statement holds for every monomorphism presentation matrix with m columns in $\mathcal{G}_Z$. Let M_f be a presentation matrix such that $|\mathcal{G}_Z| = m + 1$. Algorithm 1 performs a 'for' loop (line 4) until the 'if' statement (line 6) is true, which by Proposition 3.6 must happen before the algorithm terminates. Let r_0 be the rightmost column in $\mathcal{R}_X$ such that there is a (minimal, i.e. leftmost) $z \in \mathcal{G}_Z$ with $M_f(x, z) = 1$, where x is the row associated to r_0. Again by Proposition 3.6, column r_0 is to the right of column z. The reductions in lines 11–14 of the algorithm transform M_f into a matrix M_f^* presenting a different monomorphism $f' : Z \hookrightarrow X$. The morphism f' coincides with f on all generators of Z except for generator z', which is mapped to the nonzero element $f_{a^{z'}}(z') + f_{a^{z'}}(Z_{a^z \le a^{z'}}(z))$, where a^z and $a^{z'}$ respectively denote the degrees of z and z'. We see that f' is a monomorphism via the following pointwise argument. For every degree a, the linear function $f_a : Z_a \to X_a$ has $\ker f_a = 0$, hence it maps nonzero elements in $\{Z_{a_i^z \le a}(z_i)\}_{i=1}^m$ to linearly independent elements $\{y_j\}$ in $\operatorname{span}(X_{a_j^x \le a}(x_j))_{j=1}^n$. We see that $f_a' : Z_a \to X_a$ satisfies the same linear independence property (which implies $\ker f_a' = 0$) because the set of image elements coincides, except for possibly an element y' replaced by $y' + y$, where y is a different element of the set.

In M_f^*, the only column in $\mathcal{G}_Z$ with nonzero entry in row x is z. By removing column z and row x, we obtain a matrix with m columns in $\mathcal{G}_Z$ which is again a presentation matrix of a monomorphism. By induction hypothesis we know that the algorithm determines a function $r'_{\max} \colon \mathcal{G}_Z \backslash \{z\} \to \mathcal{R}_X$ whose image does not contain r_0 and the columns to its right. The function $r'_{\max}$ extends to a function $r_{\max} \colon \mathcal{G}_Z \to \mathcal{R}_X$ by defining $r_{\max}(z) := r_0$. Finally, we observe that the function $r_{\max}$ is injective by construction. □

Let us now go back to the reduction of presentation matrices. As with M_f, we can reduce M_b by left-to-right column transformations to get a reduced matrix $\bar{M}_b$. We denote by σ_b the permutation on $\{1, \dots, n\}$ associated with the nonzero columns of $\bar{M}_b$, which is well-defined because the matrix M_b only has columns with at most one nonzero coefficient and has the same set of columns in $\mathcal{R}_X$ as M_f. In our running example, computing $\bar{M}_b$ gives us $\sigma_b = [453261]$.

After reduction via left-to-right column operations, the matrices $\bar{M}_f$ and $\bar{M}_b$ have non-zero columns with the same set of labels, as we will prove in Proposition 3.9.

Definition 3.8 Let $n \geq 1$ be an integer and σ a permutation on $\{1, \dots, n\}$. An **inversion** of σ is a pair (i, j) of elements of $\{1, \dots, n\}$ such that $i < j$ and $\sigma(i) > \sigma(j)$.

Given a permutation σ, we also give the name **inversion** to a transposition $(i\ j)$ such that $i < j$ and $\sigma(i) < \sigma(j)$: composing σ by $(i\ j)$ on the right creates an inversion.

Using inversions we can define a poset structure on permutations: we write $\sigma \leq \sigma'$ if there exist $k \geq 0$ and a composition of transpositions $\tau = \tau_1 \cdots \tau_k$ such that $\sigma\tau = \sigma'$ and, for all $i \leq k$, τ_i is an inversion of the permutation $\sigma\tau_1 \cdots \tau_{i-1}$. In what follows, we often call τ simply a **composition of inversions of** σ when it satisfies this property. Notice that $\leq$ is a partial order on S_n, the symmetric group on $\{1, \dots, n\}$. With respect to this order, the identity permutation is the smallest element and the reverse permutation $[n\ n-1\ \dots\ 2\ 1]$ is the largest element.

Proposition 3.9 *Let $f \colon Z \hookrightarrow X$ be a monomorphism, M_f be a presentation matrix of f and M_b be the bar-to-bar matrix computed via Algorithm 1. Let $\bar{M}_f$ and $\bar{M}_b$ be reduced matrices obtained from M_f and M_b respectively, and let σ_f and σ_b be the associated permutations. Then, the following facts hold:*

- *The nonzero columns of the reduced matrices $\bar{M}_f$ and $\bar{M}_b$ are in the same positions,*
- *$\sigma_f \geq \sigma_b$, that is, $\sigma_f = \sigma_b\tau$ with τ a composition of inversions of σ_b.*

Proof Since we can replace M_f with the output M_f^* of Algorithm 1, which has the same associated permutation σ_f (as it is obtained by partially reducing M_f), and following the proof of Proposition 3.7, we can assume that M_f satisfies the following property: let z_0 be the unique column of M_f in $\mathcal{G}_Z$ such that the column $r_0 := r_{\max}(z_0)$ is maximal in the total order on columns; then the only row x_0 such that $M_f(x_0, r_{\max}(z_0)) = 1$ has exactly one other nonzero entry, which is $M_f(x_0, z_0) = 1$. We prove the claims by induction on the number of columns in $\mathcal{G}_Z$.

If $\mathcal{G}_Z = \varnothing$, then there is nothing to prove: $M_f = M_b$ and they are reduced, so $\sigma_f = \sigma_b$.

Otherwise, we execute Algorithm 1 to get the bar-to-bar matrix M_b and the function $r_{\max}$. Let z_0 be the unique column of M_f in $\mathcal{G}_Z$ such that $r_0 := r_{\max}(z_0)$ is maximal in the total order on columns. By removing column z_0, we obtain a presentation matrix

M'_f of a monomorphism f' with a set of columns $\mathcal{G}'_Z$ strictly contained in $\mathcal{G}_Z$, to which we can apply our induction hypothesis: $\bar{M}'_f$ and $\bar{M}'_b$ have the same nonzero columns, and $\sigma'_f = \sigma'_b\tau$ for some composition of inversions τ of σ'_b. The matrix M'_b, computed by using Algorithm 1 on M'_f, can be equivalently obtained by removing column z_0 from M_b, since M_f satisfies the property stated at the beginning of the proof. See Example 3.10 for matrices M'_f, M'_b, $\bar{M}'_f$ and $\bar{M}'_b$ in the running example.

Let x_0 be the only row such that $M_f(x_0, r_0) = 1$. By the execution of Algorithm 1, no other column of M'_f has a nonzero coefficient on row x_0, and so we deduce that the reductions of the matrices M'_f and M'_b do not affect column r_0. Since by inductive hypothesis $\bar{M}'_f$ and $\bar{M}'_b$ have the same nonzero columns, this implies that column r_0 does not appear in the inversions of τ, meaning that $\tau = (s_1\, t_1)(s_2\, t_2)\cdots(s_k\, t_k)$ with $s_i \neq c'_{r_0}$ and $t_i \neq c'_{r_0}$ for all $i \in \{1, \ldots, k\}$, where c'_{r_0} denotes the relative position in $\{1, \ldots, n\}$ of column r_0 in the (totally ordered) set of nonzero columns of the reduced matrix $\bar{M}'_b$.

Now, let M_g be the matrix M_f where we modify the column z_0 by setting to zero all its entries except the one on row x_0. We reduce the matrix M_f first as for M'_f, and then we reduce the column z_0 by columns to its left, which does not affect the nonzero coefficient on row x_0: we denote the resulting matrix by M''_f. $\bar{M}_f$ is then obtained by completing the reduction using column z_0. We reduce M_g and M_b in similar fashion, following M'_f and M'_b, respectively. We observe the following facts.

- The nonzero columns of $\bar{M}_f$, $\bar{M}_g$, and $\bar{M}_b$ are the nonzero columns of $\bar{M}'_f$ and $\bar{M}'_b$, except we replace r_0 with z_0. This is clear by construction for the matrices $\bar{M}_g$ and $\bar{M}_b$, as the column z_0 coincides with r_0. For the matrix $\bar{M}_f$, observe that for every nonzero entry $M_f(x, z_0)$ on column z_0, there is a nonzero entry $M_f(x, r)$ in a column r to the left of r_0, which implies that r_0 gets zeroed out after the reduction as it is linearly dependent with a number of columns to its left.
- $\sigma_f = \sigma_g\tau'$ where $\tau' := (c_{z_0}\, c_1)(c_1\, c_2)\cdots(c_{k-1}\, c_k)$ and $c_1, \ldots, c_k, c_{r_0}$ are the relative positions in $\{1, \ldots, n\}$ of the nonzero columns of M''_f whose lowest nonzero entry is modified (is moved to a different row) when reducing to $\bar{M}_f$, with c_{z_0} and c_{r_0} respectively denoting the relative positions of column z_0 and r_0 in the set of nonzero columns of M''_f.
- $\sigma_g = \sigma'_f\gamma^{-1}$ and $\sigma_b = \sigma'_b\gamma^{-1}$ where $\gamma := (c_{z_0}\, c_{z_0}+1\, \cdots\, c_{r_0})$ represents a cyclic permutation of the nonzero columns between z_0 and r_0.

See Example 3.10 for concrete examples of these relationships. We deduce that

$$\begin{aligned}\sigma_f &= \sigma_g\tau' \\ &= \sigma'_f\gamma^{-1}\tau' \\ &= \sigma'_b\tau\gamma^{-1}\tau' \\ &= \sigma_b\gamma\tau\gamma^{-1}\tau'.\end{aligned}$$

By the definition of τ', it is a composition of inversions of σ_g. We conclude the induction step by showing that $\gamma\tau\gamma^{-1}$ is a composition of inversions of σ_b.

More precisely, we know that $\tau = (s_1\, t_1) \cdots (s_k\, t_k)$ is a composition of inversions of σ'_b, meaning that $(s_i\, t_i)$ is an inversion of the permutation $\sigma'_b(s_1\, t_1) \cdots (s_{i-1}\, t_{i-1})$, for every $i \in \{1, \dots, k\}$, and we want to prove that $\gamma\tau\gamma^{-1} = (\gamma(s_1)\,\gamma(t_1)) \cdots (\gamma(s_k)\,\gamma(t_k))$ is a composition of inversions of σ_b, meaning that $(\gamma(s_i)\,\gamma(t_i))$ is an inversion of the permutation $\sigma_b(\gamma(s_1)\,\gamma(t_1)) \cdots (\gamma(s_{i-1})\,\gamma(t_{i-1}))$, for every $i \in \{1, \dots, k\}$. First, we observe that $s_i < t_i$ implies $\gamma(s_i) < \gamma(t_i)$, since as observed earlier the relative position c'_{r_0} of column r_0 in the set of nonzero columns of $\bar{M}'_b$ does not appear in τ. Let us now denote

$$
\begin{aligned}
\sigma'_{i-1} &:= \sigma'_b(s_1\, t_1) \cdots (s_{i-1}\, t_{i-1}), \\
\sigma_{i-1} &:= \sigma_b(\gamma(s_1)\,\gamma(t_1)) \cdots (\gamma(s_{i-1})\,\gamma(t_{i-1})).
\end{aligned}
$$

We have to prove that $\sigma'_{i-1}(s_i) < \sigma'_{i-1}(t_i)$ implies $\sigma_{i-1}(\gamma(s_i)) < \sigma_{i-1}(\gamma(t_i))$. This is a consequence of the equalities

$$
\sigma_{i-1}(\gamma(s_i)) = \sigma_b \gamma (s_1\, t_1) \gamma^{-1} \gamma \cdots \gamma^{-1} \gamma (s_{i-1}\, t_{i-1}) \gamma^{-1} \gamma(s_i) = \sigma'_{i-1}(s_i)
$$

and of similar equalities for t_i. □

Example 3.10 From the example matrix M_f in (5), the induction hypothesis of Proposition 3.9 with $\mathcal{G}'_Z = \mathcal{G}_Z \setminus \{z_3\}$ gives the matrices

$$
M'_f = \begin{bmatrix}
0 & & 0 & & & & 1 & \\
0 & & 0 & & & & & 1 \\
0 & & 1 & 1 & & & & \\
1 & & 0 & & 1 & & & \\
1 & 1 & 0 & & & & & \\
0 & & 0 & & & 1 & &
\end{bmatrix},\;
M'_b = \begin{bmatrix}
0 & & 0 & & & & 1 & \\
0 & & 0 & & & & & 1 \\
0 & & 1 & 1 & & & & \\
1 & & 0 & & 1 & & & \\
0 & 1 & 0 & & & & & \\
0 & & 0 & & & 1 & &
\end{bmatrix},
$$

where the column z_3 is omitted, and the reduced matrices

$$
\bar{M}'_f = \begin{bmatrix}
0 & & 0 & & & & \boxed{1} & \\
0 & & 0 & & & & & \boxed{1} \\
0 & & \boxed{1} & 0 & & & & \\
1 & \boxed{1} & 0 & & 0 & & & \\
\boxed{1} & 0 & 0 & & & & & \\
0 & & 0 & & & \boxed{1} & &
\end{bmatrix},\;
\bar{M}'_b = \begin{bmatrix}
0 & & 0 & & & & \boxed{1} & \\
0 & & 0 & & & & & \boxed{1} \\
0 & & \boxed{1} & 0 & & & & \\
\boxed{1} & & 0 & & 0 & & & \\
0 & \boxed{1} & 0 & & & & & \\
0 & & 0 & & & \boxed{1} & &
\end{bmatrix}.
$$

We find that $\sigma'_f = [543612]$ and $\sigma'_b = [453612]$, and so $\sigma'_f = \sigma'_b(1\,2)$, where $(1, 2)$ is indeed an inversion.

The induction step of Proposition 3.9 reduces the matrices

$$
M_f = \begin{bmatrix}
0 & & 0 & 1 & & & & 1 & \\
0 & & 0 & 1 & & & & & 1 \\
0 & & 1 & 0 & 1 & & & & \\
1 & & 0 & 0 & & 1 & & & \\
1 & 1 & 0 & 0 & & & & & \\
0 & & 0 & 1 & & & 1 & &
\end{bmatrix},\;
M_g = \begin{bmatrix}
0 & & 0 & 0 & & & & 1 & \\
0 & & 0 & 1 & & & & & 1 \\
0 & & 1 & 0 & 1 & & & & \\
1 & & 0 & 0 & & 1 & & & \\
1 & 1 & 0 & 0 & & & & & \\
0 & & 0 & 0 & & & 1 & &
\end{bmatrix},\;
M_b = \begin{bmatrix}
0 & & 0 & 0 & & & & 1 & \\
0 & & 0 & 1 & & & & & 1 \\
0 & & 1 & 0 & 1 & & & & \\
1 & & 0 & 0 & & 1 & & & \\
0 & 1 & 0 & 0 & & & & & \\
0 & & 0 & 0 & & & 1 & &
\end{bmatrix},
$$

to

$$
\bar{M}_f = \begin{bmatrix}
0 & & 0 & 1 & & & & \boxed{1} & \\
0 & & 0 & 1 & & & \boxed{1} & & 0 \\
0 & & \boxed{1} & 0 & 0 & & & & \\
1 & \boxed{1} & 0 & 0 & & 0 & & & \\
\boxed{1} & 0 & 0 & 0 & & & & & \\
0 & & 0 & \boxed{1} & & & 0 & &
\end{bmatrix},\;
\bar{M}_g = \begin{bmatrix}
0 & & 0 & 0 & & & & \boxed{1} & \\
0 & & 0 & \boxed{1} & & & & & 0 \\
0 & & \boxed{1} & 0 & 0 & & & & \\
1 & \boxed{1} & 0 & 0 & & 0 & & & \\
\boxed{1} & 0 & 0 & 0 & & & & & \\
0 & & 0 & 0 & & & \boxed{1} & &
\end{bmatrix},\;
\bar{M}_b = \begin{bmatrix}
0 & & 0 & 0 & & & & \boxed{1} & \\
0 & & 0 & \boxed{1} & & & & & 0 \\
0 & & \boxed{1} & 0 & 0 & & & & \\
\boxed{1} & & 0 & 0 & & 0 & & & \\
0 & \boxed{1} & 0 & 0 & & & & & \\
0 & & 0 & 0 & & & \boxed{1} & &
\end{bmatrix}.
$$

We find that $\sigma_f = [543621]$, $\sigma_g = [543261]$, and $\sigma_b = [453261]$. Thus $\sigma_f = \sigma_g(4\,5)$, $\sigma_g = \sigma'_f(4\,5\,6)$, and $\sigma_b = \sigma'_b(6\,5\,4)$.

Corollary 3.11 *Let $f\colon Z \hookrightarrow X$ be a monomorphism, and let $f_b\colon Z \hookrightarrow X$ be the associated bar-to-bar monomorphism. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ be the start-points of the bars of X, and let $b_1 \leq b_2 \leq \ldots \leq b_n$ be the degrees of the non-zero columns of $\bar{M}_f$. Then*

$$\operatorname{coker} f = \bigoplus_{j=1}^{n} K(a_j, b_{\sigma_f(j)}) \qquad \textit{and} \qquad \operatorname{coker} f_b = \bigoplus_{j=1}^{n} K(a_j, b_{\sigma_b(j)}).$$

Proof By Proposition 3.9, the real numbers $b_1 \leq b_2 \leq \ldots \leq b_n$ are also the degrees of the non-zero columns of $\bar{M}_b$. By design of the persistent homology algorithm, the barcode decomposition of coker f and coker f_b is then determined by pairing start-points $\{a_j\}$ with end-points $\{b_j\}$ following the permutations σ_f and σ_b respectively, and the claim follows. □

We state below the rearrangement inequality following (Vince 1990). Since the statement we need is slightly different from those we found in the literature, we include a short proof, which is a slight modification of the argument in (Vince 1990) and can be found also in (Steele 2004, p. 82).

Theorem 3.12 (Rearrangement inequality) *Let $g_1, g_2, \ldots, g_n$ be real valued functions defined on an interval $I \subseteq \mathbb{R}$ such that $g_{k+1} - g_k$ is a non-decreasing function, for all $k \in \{1, \ldots, n-1\}$, and let $b_1 \leq b_2 \leq \ldots \leq b_n$ be a sequence of elements of I. If $\rho \leq \sigma$ in S_n, then*

$$\sum_{k=1}^{n} g_k(b_{\rho(k)}) \geq \sum_{k=1}^{n} g_k(b_{\sigma(k)}).$$

Proof Since the argument we present can be iterated, it is enough to prove the statement for $\sigma = \rho\tau$ where $\tau = (i\ j)$ is an inversion: $i < j$ and $\rho(i) < \rho(j)$. We have

$$\begin{aligned}
\sum_{k=1}^{n} g_k(b_{\rho(k)}) - \sum_{k=1}^{n} g_k(b_{\sigma(k)}) &= g_i(b_{\rho(i)}) + g_j(b_{\rho(j)}) - g_i(b_{\sigma(i)}) - g_j(b_{\sigma(j)}) \\
&= g_i(b_{\rho(i)}) + g_j(b_{\rho(j)}) - g_i(b_{\rho(j)}) - g_j(b_{\rho(i)}) \\
&= \big(g_j(b_{\rho(j)}) - g_i(b_{\rho(j)})\big) - \big(g_j(b_{\rho(i)}) - g_i(b_{\rho(i)})\big) \geq 0,
\end{aligned}$$

where the last inequality follows from $b_{\rho(i)} \leq b_{\rho(j)}$ and from the fact that $g_j - g_i$ is non-decreasing. □

Corollary 3.13 *Let $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ be sequences of real numbers, and let $p \in [1, \infty)$. If $\rho \leq \sigma$ in S_n, then*

$$\sum_{k=1}^{n} |a_k - b_{\rho(k)}|^p \leq \sum_{k=1}^{n} |a_k - b_{\sigma(k)}|^p.$$

Proof Let $h_k(x) = |a_k - x|^p$. It is easy to check that the function $h_{k+1} - h_k$ is non-increasing for all $k \in \{1, \dots, n-1\}$, so we can apply Theorem 3.12 to the sequence of functions $g_k := -h_k$. $\square$

Theorem 3.14 *For any monomorphism $f\colon Z \hookrightarrow X$ it is possible to determine (via Algorithm* 1*) a bar-to-bar monomorphism $f_b\colon Z \hookrightarrow X$ such that $\|\operatorname{coker} f_b\|_p \leq \|\operatorname{coker} f\|_p$, for all $p \in [1, \infty]$.*

Proof First, assume $p \in [1, \infty)$. The persistence modules coker f and coker f_b have barcode decompositions as in Corollary 3.11. Then, the claim follows from Corollary 3.13 applied to the permutations $\sigma_b \leq \sigma_f$ (Proposition 3.9). The claim for $p = \infty$ follows from taking the limit for $p \to \infty$ of both sides of the inequality $\|\operatorname{coker} b\|_p \leq \|\operatorname{coker} f\|_p$, recalling that $\lim\limits_{p\to\infty} \|u\|_p = \|u\|_\infty$ for any vector $u \in \mathbb{R}^n$ (Sect. 2.6). $\square$

3.3 Bar-to-bar epimorphisms

A similar result to Theorem 3.14 exists for epimorphisms and their kernels. To show this, we use a duality argument. The dualization of persistence modules has been studied extensively, see e.g. (Bauer and Lesnick 2015; Miller 2020; Bauer and Schmahl 2023). Here we dualize tame persistence modules indexed by $[0, \infty)$, which requires some special handling since in the setting of this article bars are only supported on left-closed right-open intervals. In this subsection, we abuse the terminology introduced in Sect. 2.1 by calling bars more general persistence modules with interval support and pointwise dimension at most 1, including persistence modules supported on intervals of the form $(a, b]$ instead of $[a, b)$. To simplify the exposition, we explicitly work with finite direct sums of bars instead of general tame persistence modules, which are only equal to direct sums of bars up to isomorphism.

Definition 3.15 Let $\mathsf{Bar}_{[0,\infty)}$ be the full subcategory of tame persistence modules whose objects are finite direct sums of bars. Similarly, let $\mathsf{Bar}_{(-\infty,0]}$ be the category of finite direct sums of bars indexed by the poset $(-\infty, 0]$ with the usual order. These categories are abelian and we can represent morphisms as matrices of morphisms between summands.

We consider the contravariant functor $(-)^\vee\colon \mathsf{Bar}_{[0,\infty)} \to \mathsf{Bar}_{(-\infty,0]}$ sending an object X in $\mathsf{Bar}_{[0,\infty)}$ to the functor $X^\vee\colon (-\infty, 0] \to \mathrm{vect}_K$ defined by $X_t^\vee := \hom(X_{-t}, K)$, for all t in $(-\infty, 0]$, with transition functions $X_{s\leq t}^\vee := \hom(X_{-t\leq -s}, K)$ given by precomposition by $X_{-t\leq -s}$, for all $s \leq t$ in $(-\infty, 0]$. Similarly, a contravariant functor $\mathsf{Bar}_{(-\infty,0]} \to \mathsf{Bar}_{[0,\infty)}$ is defined, which we also denote by $(-)^\vee$ with an abuse of notation. Both functors $(-)^\vee$ are exact.

If $X = K(a, b)$, we observe that $X^\vee$ is also a bar, but its support (i.e., the set of poset elements for which the functor $X^\vee$ takes a nonzero value) is the left-open, right-closed interval $(-b, -a]$. In this article, we are considering bars whose support is a left-closed, right-open interval in $\mathbb{R}$. To fix this, we can consider the **pointwise direct limit** functor $\varinjlim_{[0,-)}(-)\colon \mathsf{Bar}_{[0,\infty)} \to \mathsf{Bar}_{[0,\infty)}$ sending X to the persistence module whose value at a is $\varinjlim_{t<a} X_t$ and whose transition functions are naturally defined. If $X = K(a, b)$, applying the pointwise direct limit functor yields the bar supported on $(a, b] \cap [0, \infty)$, whose dual $K(a, b)^\vee$ is the bar $K(-b, -a)$ in $\mathsf{Bar}_{(-\infty,0]}$, which is supported on $[-b, -a) \cap (-\infty, 0]$. Similarly, one defines the pointwise direct limit functor $\varinjlim_{(-\infty,-)}(-)\colon \mathsf{Bar}_{(-\infty,0]} \to \mathsf{Bar}_{(-\infty,0]}$. The pointwise direct limit functors are exact. Applying the functor $(-)^\vee$ after the pointwise direct limit functor is therefore an exact functor, which we denote by $(-)^\dagger$.

To summarize, we have contravariant exact functors

$$(-)^\dagger\colon \mathsf{Bar}_{[0,\infty)} \to \mathsf{Bar}_{(-\infty,0]} \quad \text{and} \quad (-)^\dagger\colon \mathsf{Bar}_{(-\infty,0]} \to \mathsf{Bar}_{[0,\infty)}$$

sending the bar $K(a, b)$ supported on $[a, b)$ to the bar $K(-b, -a)$ supported on $[-b, -a) \cap (-\infty, 0]$, and extended to the rest of the objects by additivity. These functors send morphisms to their transpose (when viewed as matrices). More precisely, given a morphism $f\colon \bigoplus_i K(a_i, b_i) \to \bigoplus_j K(c_j, d_j)$ written as the matrix $[f_{i,j}]_{i,j}$ with $f_{i,j}\colon K(a_i, b_i) \to K(c_j, d_j)$ for all i and j, the morphism $f^\dagger\colon \bigoplus_j K(-d_j, -c_j) \to \bigoplus_i K(-b_i, -a_i)$ can be written as the matrix $[f^\dagger_{j,i}]_{i,j}$.

As a consequence of the exactness of $(-)^\dagger$, for any morphism $f\colon X \to Y$ in $\mathsf{Bar}_{[0,\infty)}$ (resp. in $\mathsf{Bar}_{(-\infty,0]}$) we have

$$(\ker f)^\dagger \cong \operatorname{coker} f^\dagger \quad \text{and} \quad (\operatorname{coker} f)^\dagger \cong \ker f^\dagger.$$

Since the functors $(-)^\dagger$ send morphisms to their transpose, they send bar-to-bar morphisms to bar-to-bar morphisms. In particular, they send bar-to-bar monomorphisms to bar-to-bar epimorphisms, and vice-versa. It is also clear that the functors $(-)^\dagger$ preserve p-norms: $\|X^\dagger\|_p = \|X\|_p$.

Moreover, we can apply the theory of Sect. 3.2 to the category $\mathsf{Bar}_{(-\infty,0]}$, and in particular apply Theorem 3.14. We conclude with the following result:

Theorem 3.16 *For any epimorphism $f\colon Z \twoheadrightarrow X$ of persistence modules, it is possible to determine a bar-to-bar epimorphism $f_b\colon Z \twoheadrightarrow X$ such that $\|\ker f_b\|_p \le \|\ker f\|_p$, for all $p \in [1, \infty]$.*

Proof As for the case of monomorphisms, we assume that X and Z are finite direct sums of finite bars. We apply Theorem 3.14 to $f^\dagger\colon X^\dagger \hookrightarrow Z^\dagger$ to get a bar-to-bar monomorphism g. We set $f_b := g^\dagger\colon Z \twoheadrightarrow X$ a bar-to-bar epimorphism, and we observe that

$$\|\ker f_b\|_p = \|\operatorname{coker} g\|_p \le \|\operatorname{coker} f^\dagger\|_p = \|\ker f\|_p.$$

□

In the case $p = \infty$, a result similar to Theorems 3.14 and 3.16 based on induced matchings (see Sect. 3.4) instead of bar-to-bar monomorphisms and epimorphisms is proved in (Bauer and Lesnick 2020, Theorem 2).

3.4 Comparing bar-to-bar morphisms with induced matchings

In Bauer and Lesnick (2015) the authors introduce a construction similar to bar-to-bar morphisms, namely **induced matchings**. In particular, given persistence modules X and Z such that there exists a monomorphism $Z \hookrightarrow X$, the authors define a **canonical injection** from the multiset of bars of Z to the multiset of bars of X, where for all real numbers $b \in [0, \infty]$ the i^{th} longest bar of Z with end-point b is sent to the i^{th} longest bar of X with the same end-point. This injection induces a bar-to-bar monomorphism, which we define here:

Definition 3.17 Let Z and X be persistence modules such that there exists a monomorphism $Z \hookrightarrow X$, and fix barcode decompositions $Z = \bigoplus_{i=1}^{m} K(a_i^z, b_i^z)$ and $X = \bigoplus_{j=1}^{n} K(a_j^x, b_j^x)$. The induced matching of Bauer and Lesnick (2015) then corresponds to a canonical injection $\varphi\colon \{1, \dots, m\} \hookrightarrow \{1, \dots, n\}$ such that $a_i^z \geq a_{\varphi(i)}^x$ and $b_i^z = b_{\varphi(i)}^x$, for all $i \in \{1, \dots, m\}$.

We define the **monomorphism induced by the canonical injection** φ as the monomorphism $f_\varphi\colon Z \hookrightarrow X$ given by

$$f_\varphi = \bigoplus_{i=1}^{m} (K(a_i^z, b_i^z) \hookrightarrow K(a_{\varphi(i)}^x, b_{\varphi(i)}^x)) \oplus \bigoplus_{j \in \{1,\dots,n\} \setminus \operatorname{im} \varphi} (0 \hookrightarrow K(a_j^x, b_j^x)).$$

Note that this is a bar-to-bar monomorphism.

Remark 3.18 Let $f\colon Z \hookrightarrow X$ be a monomorphism. If the bars of X all have distinct end-points, then the monomorphism induced by the canonical injection (Definition 3.17) coincides, up to isomorphism, with the bar-to-bar monomorphism f_b as determined in Sect. 3.2. This is because, in this case, there is only one bar-to-bar monomorphism from Z to X (up to isomorphism).

Remark 3.19 In general, this is not necessarily the case. For example, starting from the monomorphism $f\colon K(2, 3) \hookrightarrow K(1, 3) \oplus K(2, 3)$ defined by

$$f = (K(2, 3) \hookrightarrow K(2, 3)) \oplus (0 \hookrightarrow K(1, 3)),$$

then we have $f_\varphi = (K(2, 3) \hookrightarrow K(1, 3)) \oplus (0 \hookrightarrow K(2, 3))$, while $f_b = f$.

More generally, monomorphisms induced by canonical injections do not commute with direct sums of monomorphisms (Bauer and Lesnick 2015, Example 5.8), while the bar-to-bar monomorphisms we introduced in Sect. 3.2 do, as an immediate consequence of their definition via Algorithm 1. That is, given $f\colon Z \hookrightarrow X$ and $f'\colon Z' \hookrightarrow X'$, it is not true in general that $(f \oplus f')_\varphi = f_\varphi \oplus f'_\varphi$, while $(f \oplus f')_b = f_b \oplus f'_b$ holds.

Maintaining a close relation between a given monomorphism $f\colon Z \hookrightarrow X$ and the bar-to-bar monomorphism f_b is instrumental to proving Proposition 3.9, the key technical result of Sect. 3.2, and in turn Theorem 3.14 and the dual Theorem 3.16. Since the canonical injection (Bauer and Lesnick 2015) does not depend on the specific given monomorphism between Z and X, but just on the existence of such a monomorphisms, it is not possible to prove Proposition 3.9 using the same strategy with the canonical injection instead of the bar-to-bar monomorphism we introduced.

We conclude this section with two corollaries to Theorems 3.14 and 3.16. Using Theorem 3.14, the problem of minimizing the p-norm of cokernels of all possible monomorphisms between two given persistence modules can be solved by restricting to bar-to-bar monomorphisms. This simplification allows for an easy combinatorial proof of the fact that, among all monomorphisms, the one induced by the canonical injection minimizes the p-norm of the cokernel. A dual result holds for epimorphisms.

Corollary 3.20 *Let Z and X be two persistence modules such that there exists a monomorphism $Z \hookrightarrow X$. Then $\min_{f\colon Z\hookrightarrow X} \|\operatorname{coker} f\|_p = \|\operatorname{coker} f_\varphi\|_p$ for all $p \in [1,\infty]$, where f_φ is the bar-to-bar monomorphism induced by the canonical injection (Definition 3.17).*

Proof By Theorem 3.14, it suffices to show the inequality $\|\operatorname{coker} f_\varphi\|_p \le \|\operatorname{coker} f\|_p$ for bar-to-bar monomorphisms $f\colon Z \hookrightarrow X$. Let $\{b_k\}_{k=1}^{\ell}$ be the set of distinct end-points of X. Then, by hypothesis that there exists a monomorphism $Z \hookrightarrow X$, we can decompose $Z = \bigoplus_{k=1}^{\ell} Z^{(k)}$ and $X = \bigoplus_{k=1}^{\ell} X^{(k)}$ where $Z^{(k)}$ (resp. $X^{(k)}$) is the direct sum of the bars in Z (resp. X) with end-point b_k. Given a bar-to-bar monomorphism $f\colon Z \hookrightarrow X$, this induces monomorphisms $f^{(k)}\colon Z^{(k)} \hookrightarrow X^{(k)}$. We then observe that $f = \bigoplus_{k=1}^{\ell} f^{(k)}$, and so

$$\|\operatorname{coker} f\|_p = \left\| \left(\|\operatorname{coker} f^{(k)}\|_p \right)_{k=1,\dots,\ell} \right\|_p .$$

Since p-norms are nondecreasing with respect to the coordinate-wise order on $[0,\infty)^{\ell}$ (Sect. 2.6), proving that $\|\operatorname{coker} f_\varphi^{(k)}\|_p \le \|\operatorname{coker} f^{(k)}\|_p$ for each $k \in \{1,\dots,\ell\}$ implies that $\|\operatorname{coker} f_\varphi\|_p \le \|\operatorname{coker} f\|_p$.

Thus it suffices to prove the result in the case where the bars of Z and X all have the same end-point, which is what we assume for the rest of the proof. Denote by b_0 this common end-point and write $Z = \bigoplus_{i=1}^{m} K(a_i^z, b_0)$ and $X = \bigoplus_{j=1}^{n} K(a_j^x, b_0)$, where the a_i^z and a_j^x are in nondecreasing order. Define $a_i^z = b_0$ for $i \in \{m+1,\dots,n\}$. Then every bar-to-bar monomorphism $f\colon Z \hookrightarrow X$ has the form

$$f = \bigoplus_{i=1}^{n} (K(a_i^z, b_0) \hookrightarrow K(a_{\alpha(i)}^x, b_0)),$$

where $\alpha\colon \{1,\dots,n\} \to \{1,\dots,n\}$ is a permutation and $K(b_0,b_0)$ denotes the zero module. By Remark 3.2,

$$\operatorname{coker} f = \bigoplus_{i=1}^{n} K\left(a_{\alpha(i)}^x, a_i^z\right).$$

In particular, the bar-to-bar monomorphism induced by the canonical injection corresponds to the permutation $\alpha = \mathrm{id}$. For all $p \in [1,\infty)$, we then apply the rearrangement inequality of Corollary 3.13 to deduce that $\| \operatorname{coker} f_\varphi \|_p \le \| \operatorname{coker} f \|_p$. The case $p = \infty$ follows by taking the limit (as in the proof of Theorem 3.14). □

We now dualize the definitions and results for the case of epimorphisms. An epimorphism $f : Z \twoheadrightarrow X$ induces a canonical injection (Bauer and Lesnick 2015) from the multiset of bars of X to the multiset of bars of Z, where for all $a \in [0, \infty]$, the i^{th} longest bar of X with start-point a is sent to the i^{th} longest bar of Z with the same start-point.

Definition 3.21 Let Z and X be persistence modules such that there exists an epimorphism $Z \twoheadrightarrow X$, and fix barcode decompositions $Z = \bigoplus_{i=1}^m K(a_i^z, b_i^z)$ and $X = \bigoplus_{j=1}^n K(a_j^x, b_j^x)$. The induced matching of Bauer and Lesnick (2015) then corresponds to a canonical injection $\psi : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, m\}$ such that $a_j^x = a_{\psi(j)}^z$ and $b_j^x \le b_{\psi(j)}^z$, for all $j \in \{1, \ldots, n\}$.

We define the **epimorphism induced by the canonical injection** ψ as the epimorphism $f_\psi : Z \twoheadrightarrow X$ given by

$$f_\psi = \bigoplus_{j=1}^n \left(K\left(a_{\psi(j)}^z, b_{\psi(j)}^z\right) \twoheadrightarrow K\left(a_j^x, b_j^x\right)\right) \oplus \bigoplus_{i \in \{1,\ldots,m\} \setminus \operatorname{im} \psi} \left(K\left(a_i^x, b_i^x\right) \twoheadrightarrow 0\right).$$

As in the case of monomorphisms, given an epimorphism $f : Z \twoheadrightarrow X$, the associated bar-to-bar epimorphisms f_ψ and f_b (see Sect. 3.3) are not necessarily the same.

Corollary 3.22 *Let Z and X be two persistence modules such that there exists an epimorphism $Z \twoheadrightarrow X$. Then $\min_{f : Z \twoheadrightarrow X} \| \ker f \|_p = \| \ker f_\psi \|_p$ for all $p \in [1, \infty]$, where f_ψ is a bar-to-bar epimorphism induced by the canonical injection (Definition* 3.21*).*

We omit the proof of Corollary 3.22, which is based on Theorem 3.16 and analogous to the proof of Corollary 3.20.

4 Noise systems and Wasserstein pseudometrics

In this section we study algebraic Wasserstein pseudometrics between persistence modules. After introducing in Sect. 4.1 a generalization of the pseudometrics associated with a noise system, we study in Sect. 4.2 noise systems determined by p-norms of persistence modules and regular contours. Section 4.3 is devoted to the associated algebraic Wasserstein pseudometrics. For some choices of parameters, these pseudometrics have a combinatorial interpretation, as we show in Sect. 4.4. Finally, in Sect. 4.5 we present formulas to compute the algebraic Wasserstein pseudometric between persistence modules in some specific cases.

4.1 Pseudometrics associated to noise systems

Given a noise system $\mathcal{S}$ and $p \in [1, \infty]$, in this section we will introduce pseudometrics $d_{\mathcal{S}}^p$ between persistence modules. These pseudometrics are a simple generalization to

$p > 1$ of the pseudometric associated to a noise system in Scolamiero et al. (2017) (see Sect. 2.4), where $p = 1$. Although the statements in this section hold true for tame functors indexed by $[0, \infty)^r$ for every positive natural number r, as in Scolamiero et al. (2017), we will limit the presentation to $r = 1$, since this is the setting of the following sections.

Definition 4.1 Let X and Y be persistence modules. A **span** of X, Y is a triplet (Z, f, g) with Z a persistence module and $f : Z \to X$ and $g : Z \to Y$ morphisms between persistence modules. A span of X, Y is therefore a diagram in Tame of the form

$$X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y.$$

Definition 4.2 Let X and Y be persistence modules, and let $\mathcal{S}$ be a noise system. A span $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ is called a $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$**-span** if

$$\ker f \in \mathcal{S}_{\varepsilon_1}, \quad \operatorname{coker} f \in \mathcal{S}_{\varepsilon_2}, \quad \ker g \in \mathcal{S}_{\varepsilon_3} \quad \text{and} \quad \operatorname{coker} g \in \mathcal{S}_{\varepsilon_4}.$$

Definition 4.3 Let X and Y be persistence modules, and let $\mathcal{S}$ be a noise system. For $p \in [1, \infty]$ and $\varepsilon \in [0, \infty)$, we say that X and Y are ε-**close in** p**-norm** $\|\cdot\|_p$ if there exists a $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$-span $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, \infty)$ such that $\|(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|_p \leq \varepsilon$. We define

$$d^p_{\mathcal{S}}(X, Y) := \inf \{\varepsilon \in [0, \infty) \mid X \text{ and } Y \text{ are } \varepsilon\text{-close in } p\text{-norm}\},$$

adopting the convention $\inf \varnothing = \infty$.

Our next aim is to prove that $d^p_{\mathcal{S}}$ is a pseudometric on Tame. We start by generalizing Proposition 8.5 in Scolamiero et al. (2017) to our current framework. Even if the generalization is not difficult, we include the proof to highlight how the properties of p-norms on $\mathbb{R}^4$ are used. We note that a similar result can be obtained for a larger family of subadditive functions on $\mathbb{R}^4$ which include p-norms (see Giunti et al. 2024, Sect. 2.1).

Proposition 4.4 *Let F, G, H be persistence modules. Assume that F and G are ε-close in p-norm, and that G and H are τ-close in p-norm. Then F and H are $(\varepsilon + \tau)$-close in p-norm.*

Proof By assumption there exists a $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$-span $F \overset{f'}{\longleftarrow} X \overset{f''}{\longrightarrow} G$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0, \infty)$ such that $\|(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|_p \leq \varepsilon$ and a $(\tau_1, \tau_2, \tau_3, \tau_4)$-span $G \overset{g'}{\leftarrow} Y \overset{g''}{\longrightarrow} H$ with $\tau_1, \tau_2, \tau_3, \tau_4 \in [0, \infty)$ such that $\|(\tau_1, \tau_2, \tau_3, \tau_4)\|_p \leq \tau$. Consider the following diagram, where the square is a pullback:

$$\begin{tikzcd}
 & & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
 & X \arrow[dl, "f'"'] \arrow[dr, "f''"] & & Y \arrow[dl, "g'"'] \arrow[dr, "g''"] & \\
F & & G & & H
\end{tikzcd}$$

By Scolamiero et al. (2017, Proposition 8.1), ker $f \in \mathcal{S}_{\tau_1}$ and coker $f \in \mathcal{S}_{\tau_2}$, hence by Scolamiero et al. (2017, Proposition 8.2) ker $f'f \in \mathcal{S}_{\varepsilon_1+\tau_1}$ and coker $f'f \in \mathcal{S}_{\varepsilon_2+\tau_2}$. By a similar argument, ker $g''g \in \mathcal{S}_{\varepsilon_3+\tau_3}$ and coker $g''g \in \mathcal{S}_{\varepsilon_4+\tau_4}$. This proves that F and H are η-close in p-norm, where $\eta := \|(\varepsilon_1+\tau_1, \varepsilon_2+\tau_2, \varepsilon_3+\tau_3, \varepsilon_4+\tau_4)\|_p$. Our claim follows from the inequality

$$\begin{aligned}\|(\varepsilon_1+\tau_1, \varepsilon_2+\tau_2, \varepsilon_3+\tau_3, \varepsilon_4+\tau_4)\|_p &\leq \|(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)\|_p + \|(\tau_1, \tau_2, \tau_3, \tau_4)\|_p \\ &\leq \varepsilon + \tau,\end{aligned}$$

which expresses the subadditivity of $\|\cdot\|_p$ and the hypotheses. □

We are now ready to prove that $d^p_{\mathcal{S}}$ is a pseudometric on Tame.

Proposition 4.5 *Given $p \in [1, \infty]$ and a noise system $\mathcal{S}$, the function $d^p_{\mathcal{S}}$ in Definition 4.3 is a pseudometric on* Tame *(see Sect. 2.4).*

Proof If $g : X \to Y$ is an isomorphism of persistence modules, the span $X \xleftarrow{\mathrm{id}} X \xrightarrow{g} Y$ shows that $d^p_{\mathcal{S}}(X, Y) = 0$. For all persistence modules X and Y, the bijection between spans $X \xleftarrow{f} Z \xrightarrow{g} Y$ between X and Y and spans $Y \xleftarrow{g} Z \xrightarrow{f} X$ between Y and X implies that $d^p_{\mathcal{S}}(X, Y) = d^p_{\mathcal{S}}(Y, X)$. Proposition 4.4 shows that the triangle inequality holds true. □

Remark 4.6 Given a noise system $\mathcal{S}$, the pseudometrics $d^p_{\mathcal{S}}$ for all $p \in [1, \infty]$ are strongly equivalent. Assuming $p \leq q$, for any pair of persistence modules X, Y we have

$$d^q_{\mathcal{S}}(X, Y) \leq d^p_{\mathcal{S}}(X, Y) \leq 4^{\left(\frac{1}{p}-\frac{1}{q}\right)} d^q_{\mathcal{S}}(X, Y),$$

as can be easily concluded from the properties on p-norms on $\mathbb{R}^4$ stated in Sect. 2.6.

4.2 *p*-norms of persistence modules and contours

The aim of this section is to introduce and study a generalization of the notion of p-norm of a persistence module (see Sect. 2.6) first introduced in Skraba and Turner (2020), that coincides with the original definition if C is the standard contour (see Sect. 2.2).

Definition 4.7 Let C be a regular contour. For $p \in [1, \infty]$, define the (p, C)**-norm** of a persistence module $X \cong \bigoplus_{i=1}^{k} K(a_i, b_i)$ by

$$\|X\|_{p,C} := \begin{cases} \left(\sum_{i=1}^{k} \ell(a_i, b_i)^p\right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \max\{\ell(a_i, b_i)\}_{i=1}^{k} & \text{for } p = \infty, \end{cases}$$

where $\ell(a_i, b_i)$ denotes the lifetime of the bar $K(a_i, b_i)$ with respect to C (see Sect. 2.2).

We see that $\|X\|_{p,C}$ does not depend on the choice of barcode decomposition for X. For $p \in [1, \infty]$ and $\varepsilon \in [0, \infty)$, consider the class of tame persistence modules

$$\mathcal{S}_{\varepsilon}^{p,C} := \{X \in \mathsf{Tame} \mid \|X\|_{p,C} \leq \varepsilon\},$$

and denote $\mathcal{S}^{p,C} := \{\mathcal{S}_{\varepsilon}^{p,C}\}_{\varepsilon \in [0,\infty)}$. If D is the standard contour (see Sect. 2.2), then $\ell(a_i, b_i) = b_i - a_i$ and we have $\|X\|_{p,D} = \|X\|_p$ and $\mathcal{S}^{p,D} = \mathcal{S}^p$. The main result in this subsection is showing that $\mathcal{S}^{p,C}$ is a noise system (see Sect. 2.3) whenever C is an action, for any $p \in [1, \infty]$. For the standard contour, this result together with Proposition 4.5 provide an alternative proof to the one in Skraba and Turner (2020) that the algebraic p-Wasserstein distance is a pseudometric, as will be later highlighted in Remark 4.20.

Given a contour C, the function $C(0, -)\colon [0, \infty) \to [0, \infty)$ is nondecreasing (and it is an increasing bijection if the contour is regular). Hence it can be viewed as a functor from the poset category $[0, \infty)$ to itself, that can be effectively used to re-parameterize $[0, \infty)$. For any persistence module X, the composition of functors $T_C(X) := XC(0, -)\colon [0, \infty) \to \mathrm{vect}_K$ is a persistence module. As we will show, $T_C(X)$ is in Tame whenever X is in Tame and C is a regular contour (Corollary 4.9). The assignment $X \mapsto T_C(X)$ can be extended to a functor $T_C\colon \mathsf{Tame} \to \mathsf{Tame}$ sending a morphism $f\colon X \to Y$ of persistence modules to the morphism $T_C(f)\colon T_C(X) \to T_C(Y)$ defined as the natural transformation between $T_C(X)$ and $T_C(Y)$ whose component at $a \in [0, \infty)$ is $T_C(f)_a = f_{C(0,a)}\colon X_{C(0,a)} \to Y_{C(0,a)}$.

We now explain the relationship between the barcode decompositions of X and $T_C(X)$ when C is a regular contour.

Proposition 4.8 *Let C be a regular contour, and let ℓ be the associated lifetime function. Consider a bar $K(a, b)$. Then*

$$T_C(K(a, b)) \cong K(\ell(0, a), \ell(0, b)).$$

Proof The functor $T_C(K(a, b))\colon [0, \infty) \to \mathrm{vect}_K$ sends $c \leq d$ in $[0, \infty)$ to the linear function

$$K(a, b)_{C(0,c) \leq C(0,d)}\colon K(a, b)_{C(0,c)} \to K(a, b)_{C(0,d)},$$

which is the identity on K if $a \leq C(0,c) \leq C(0,d) < b$ and the zero function otherwise. Since C is regular, $\ell(0,-)$ is a strictly increasing function, hence the condition $a \leq C(0,c) \leq C(0,d) < b$ is equivalent to $\ell(0,a) \leq c \leq d < \ell(0,b)$. $\square$

Corollary 4.9 *Let X be a tame persistence module with barcode decomposition $\bigoplus_{i=1}^{k} K(a_i,b_i)$, and let C be a regular contour. Then $T_C(X) \cong \bigoplus_{i=1}^{k} K(\ell(0,a_i), \ell(0,b_i))$. In particular $T_C(X)$ is also in* Tame.

Proof Since direct sums in Tame are defined pointwise (Sect. 2.1), if $\{X_i\}_{i\in I}$ is a finite collection of persistence modules and C is a regular contour, then $T_C(\bigoplus_{i\in I} X_i) \cong \bigoplus_{i\in I} T_C(X_i)$, this together with Proposition 4.8, gives the following.

$$\begin{aligned} T_C(X) &\cong T_C\left(\bigoplus_{i=1}^{k} K(a_i,b_i)\right) && \text{(by Proposition 4.8)} \\ &\cong \bigoplus_{i=1}^{k} T_C\left(K(a_i,b_i)\right) \\ &\cong \bigoplus_{i=1}^{k} K\left(\ell(0,a_i), \ell(0,b_i)\right). \end{aligned}$$

Given that bars $K(\ell(0,a_i), \ell(0,b_i))$ are in Tame and tameness is preserved by finite direct sums, $T_C(X)$ is also in Tame. $\square$

We now show that the functor T_C is exact.

Proposition 4.10 *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* Tame*, and let C be a regular contour. Then the sequence $0 \to T_C(X) \to T_C(Y) \to T_C(Z) \to 0$ is also exact in* Tame.

Proof Exactness in Tame is defined pointwise: $0 \to X \to Y \to Z \to 0$ is exact if and only if $0 \to X_a \to Y_a \to Z_a \to 0$ is exact in vect_K, for every $a \in [0,\infty)$. As a consequence, $0 \to X_{C(0,b)} \to Y_{C(0,b)} \to Z_{C(0,b)} \to 0$ is exact in vect_K, for every $b \in [0,\infty)$, hence by definition the sequence $0 \to T_C(X) \to T_C(Y) \to T_C(Z) \to 0$ is exact. The fact that the exact sequence $0 \to T_C(X) \to T_C(Y) \to T_C(Z) \to 0$ is in Tame follows from Corollary 4.9 and how T_C is defined on morphisms. $\square$

Remark 4.11 As is clear from its proof, Proposition 4.10 in fact holds for the precomposition of persistence modules by any increasing bijection of $[0,\infty)$.

In the rest of the article, we will focus on contours that are regular and actions (see Sect. 2.2), called regular actions for brevity. We prove here a simple but important property of regular actions, and the associated lifetime function ℓ, which is used to prove the subsequent results.

Lemma 4.12 *If C is a regular action, then $\ell(a,c) = \ell(a,b)+\ell(b,c)$ for any $a \leq b \leq c$ in $[0,\infty)$.*

Proof Let $a \leq b \leq c$. Using the definitions and the assumption that C is an action, we have $C(C(a, \ell(a,b)), \ell(b,c)) = C(a, \ell(a,b) + \ell(b,c))$. Again by definition, we observe that the left-hand side equals c, and that $c = C(a, \ell(a,b) + \ell(b,c))$ implies $\ell(a,c) = \ell(a,b) + \ell(b,c)$. □

Proposition 4.13 *Let X be a persistence module, let $p \in [1, \infty]$, and let C be a regular action. Then $\|X\|_{p,C} = \|T_C(X)\|_p$.*

Proof Let $X \cong \bigoplus_{i=1}^k K(a_i, b_i)$. For any fixed $p \in [1, \infty)$, we have

$$\begin{aligned}
\|T_C(X)\|_p &= \left(\sum_{i=1}^k (\ell(0, b_i) - \ell(0, a_i))^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{i=1}^k \ell(a_i, b_i)^p\right)^{\frac{1}{p}} \\
&= \|X\|_{p,C},
\end{aligned}$$

where the first equality is by Corollary 4.9, the second one is by Lemma 4.12, and the third one is by definition of $\|\cdot\|_{p,C}$. The case $p = \infty$ is similar. □

We are now ready to prove that $\mathcal{S}^{p,C}$, with C a regular action, satisfies the axioms in the definition of noise system (see Sect. 2.3).

Lemma 4.14 *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* Tame*, and let C be a regular contour. Then $\|X\|_{p,C} \leq \|Y\|_{p,C}$ and $\|Z\|_{p,C} \leq \|Y\|_{p,C}$.*

For the standard contour, our statement coincides with Lemma 7.8 in Skraba and Turner (2020), which is easily proven using the induced matchings (Bauer and Lesnick 2015) for monomorphisms and epimorphisms of persistence modules. For the sake of completeness, we include the proof for $\|\cdot\|_{p,C}$, which does not present any additional difficulty.

Proof The existence of a monomorphism from X to Y implies the existence of the bar-to-bar monomorphism $f_\varphi \colon X \hookrightarrow Y$ of Definition 3.17, induced by the canonical injection (Bauer and Lesnick 2015). The monomorphism f_φ decomposes as a finite direct sum of monomorphisms of the form $K(a', b) \hookrightarrow K(a, b)$, with $a \leq a'$, and of the form $0 \hookrightarrow K(a, b)$. By monotonicity of contours, $a \leq a'$ implies $\ell(a', b) \leq \ell(a, b)$. The inequality $\|X\|_{p,C} \leq \|Y\|_{p,C}$ then follows from the definition of $\|\cdot\|_{p,C}$, since every term in the expression for $\|X\|_{p,C}$ is upper bounded by a term in the expression for $\|Y\|_{p,C}$.

The proof of the inequality $\|Z\|_{p,C} \leq \|Y\|_{p,C}$ is obtained similarly, using the epimorphism induced by the canonical injection (see Definition 3.21). □

Lemma 4.15 *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in* Tame*, and let C be a regular action. Then $\|Y\|_{p,C} \leq \|X\|_{p,C} + \|Z\|_{p,C}$.*

Proof First, we prove the statement assuming that C is the standard contour. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence of persistence modules, and let us show that $\|Y\|_p \le \|X\|_p + \|Z\|_p$. We consider the monomorphism f and observe that $Z \cong \operatorname{coker} f$ implies that Z and $\operatorname{coker} f$ have the same barcode decomposition, hence $\|Z\|_p = \|\operatorname{coker} f\|_p$. Theorem 3.14 tells us that, among all monomorphisms between two fixed persistence modules, the norm $\|\cdot\|_p$ of the cokernel is minimized by a bar-to-bar monomorphism. We therefore just need to prove that $\|Y\|_p \le \|X\|_p + \|\operatorname{coker} f\|_p$, for any bar-to-bar monomorphism f between X and Y.

By Remark 3.2, if $f : X \to Y$ is a bar-to-bar monomorphism, then there exist barcode decompositions $\bigoplus_{i=1}^m X_i$ and $\bigoplus_{j=1}^n Y_j$ of X and Y, respectively, such that $m \le n$ and, up to permutation of the Y_j, there are monomorphisms $f_i : X_i \to Y_i$ between bars such that $\operatorname{coker} f = \bigoplus_{i=1}^m \operatorname{coker} f_i \oplus \bigoplus_{j=m+1}^n Y_j$. We observe that, for each bar $Y_i = K(a_i, b_i)$ of Y with $i \in \{1, \dots, m\}$, there is a bar $X_i = K(a_i', b_i)$ of X and a corresponding summand $\operatorname{coker} f_i$ of $\operatorname{coker} f$, which is a bar $K(a_i, a_i')$ if $a_i < a_i'$, and it is the zero module if $a_i = a_i'$. Similarly, we observe that each bar $Y_j = K(a_j, b_j)$ of Y with $j \in \{m+1, \dots, n\}$ is also a bar of $\operatorname{coker} f$. By definition, $\|Y\|_p$ is the p-norm of the following element of $\mathbb{R}^n$:

$$(b_j - a_j)_{j \in \{1,\dots,n\}} = (((b_i - a_i') + (a_i' - a_i))_{i \in \{1,\dots,m\}}, (b_j - a_j)_{j \in \{m+1,\dots,n\}}).$$

Then, by the triangular inequality of p-norms in $\mathbb{R}^n$, we have $\|Y\|_p \le \|X\|_p + \|\operatorname{coker} f\|_p$, which completes the proof when C is the standard contour.

Let now C be any regular action. By Proposition 4.10, exactness of $0 \to X \to Y \to Z \to 0$ implies exactness of $0 \to T_C(X) \to T_C(Y) \to T_C(Z) \to 0$. Applying the previous part of the proof to the latter exact sequence yields $\|T_C(Y)\|_p \le \|T_C(X)\|_p + \|T_C(Z)\|_p$, which by Proposition 4.13 coincides with our claim. □

For the standard contour, the statement of Lemma 4.15 is given in Remark 7.32 of Skraba and Turner (2020). However, to our knowledge, we provide the first proof of this inequality that does not assume the fact that the p-norm of persistence modules induces a pseudometric. Indeed in Skraba and Turner (2020) the fact that the algebraic Wasserstein distance satisfies the triangular inequality is used as an hypothesis.

We can now prove the main result of this subsection.

Theorem 4.16 *For any $p \in [1, \infty]$ and any regular action C, $\mathcal{S}^{p,C}$ is a noise system.*

Proof We show that $\mathcal{S}^{p,C}$ satisfies all axioms of the definition of noise system (see Sect. 2.3). Since the norm $\|\cdot\|_{p,C}$ of the zero module 0 is zero, we have $0 \in \mathcal{S}_\varepsilon^{p,C}$, for all $\varepsilon \in [0, \infty)$. By definition of $\mathcal{S}^{p,C}$, it is clear that $\mathcal{S}_\tau^{p,C} \subseteq \mathcal{S}_\varepsilon^{p,C}$ whenever $\tau \le \varepsilon$. Lemmas 4.14 and 4.15 complete the proof, showing that $\mathcal{S}^{p,C}$ satisfies both conditions on short exact sequences of persistence modules. □

Remark 4.17 For $p < \infty$, the noise system $\mathcal{S}^{p,C}$ in not closed under direct sums (Sect. 2.3), since $\|X \oplus Y\|_{p,C} = \left\| \left(\|X\|_{p,C}, \|Y\|_{p,C} \right) \right\|_p$ by Eq. (2).

Remark 4.18 Let us briefly highlight the role of our hypotheses on contours, which are required to be regular actions in Theorem 4.16. The regularity assumption ensures for instance that the associated lifetime function ℓ is well-defined, and that the functor T_C is an endofunctor on Tame. The assumption that $C(0,-)\colon [0,\infty) \to [0,\infty)$ is an increasing bijection is sufficient to prove many results of this subsection (see Remark 4.11), but we choose to assume the stronger condition of regularity on C to facilitate a comparison with the results of Chachólski and Riihimäki (2020), observing in addition that many examples of regular contours can be found, for example the contours of distance type (Sect. 2.2) that are used in our experiments (see Sect. 5). The hypothesis that the considered contours are actions is motivated by the use of Proposition 4.13 in the proof of Lemma 4.15. If C is not an action, the equalities of Lemma 4.12 and Proposition 4.13 are replaced by the inequalities $\ell(a,c) \le \ell(a,b) + \ell(b,c)$, for any $a \le b \le c$, and $\|T_C(X)\|_p \le \|X\|_{p,C}$. A proof of Lemma 4.15 removing the action hypothesis on C eludes us.

4.3 Contours and algebraic Wasserstein distances

We now turn to considering the pseudometrics $d^q_{\mathcal{S}^{p,C}}$ associated (as in Sect. 4.1) with the noise systems $\mathcal{S}^{p,C}$ introduced in Sect. 4.2, for fixed $p, q \in [1,\infty]$ and a regular action C. We also refer to these pseudometrics as **algebraic Wasserstein distances**. First, we show that the functor T_C introduced in Sect. 4.2 allows us to switch between a pseudometric $d^q_{\mathcal{S}^{p,C}}$ and the pseudometric $d^q_{\mathcal{S}^p}$ associated with the standard contour. More precisely, we show that T_C can be viewed as an isometry

$$T_C\colon \left(\mathsf{Tame}, d^q_{\mathcal{S}^{p,C}}\right) \to \left(\mathsf{Tame}, d^q_{\mathcal{S}^p}\right).$$

Let us recall that, if C is a regular contour, the function $C(0,-) : [0,\infty) \to [0,\infty)$ is an increasing bijection. Its inverse $\ell(0,-) := C(0,-)^{-1}$ is therefore an increasing bijection as well. Mimicking the definition of T_C given in Sect. 4.2, we can define a functor $T_\ell\colon \mathsf{Tame} \to \mathsf{Tame}$ given by precomposition by the increasing function $\ell(0,-)$. By Proposition 4.10, the functor $T_C\colon \mathsf{Tame} \to \mathsf{Tame}$ preserves kernels and cokernels, and T_ℓ has the same property by Remark 4.11. Furthermore, since $C(0,-)$ and $\ell(0,-)$ are inverse to each other, the compositions $T_C T_\ell$ and $T_\ell T_C$ are the identity functor 1_{Tame} on Tame.

To prove the following result, it is convenient to define the (p,q,C)**-cost** of a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ of persistence modules as the element $c \in [0,\infty]$ defined by

$$c := \left\| \left(\|\ker f\|_{p,C}, \|\operatorname{coker} f\|_{p,C}, \|\ker g\|_{p,C}, \|\operatorname{coker} g\|_{p,C} \right) \right\|_q .$$

Proposition 4.19 *Let C be a regular action, and let X, Y be persistence modules. Then*

$$d^q_{\mathcal{S}^{p,C}}(X,Y) = d^q_{\mathcal{S}^p}(T_C(X), T_C(Y)).$$

Proof Let D denote the standard contour, and let us recall that the (p,D)-norm of a persistence module coincides with its p-norm (Sect. 4.2). We describe a correspondence between spans having the same cost, calculated with respect to (p,q,C) and (p,q,D) respectively.

Let $X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y$ be a span and let c be its (p, q, C)-cost. Applying the functor T_C, we obtain the span $T_C(X) \xleftarrow{T_C(f)} T_C(Z) \xrightarrow{T_C(g)} T_C(Y)$, whose (p, q, D)-cost is

$$\begin{aligned} c' &= \left\| \left(\|\ker T_C(f)\|_p \,, \|\mathrm{coker}\, T_C(f)\|_p \,, \|\ker T_C(g)\|_p \,, \|\mathrm{coker}\, T_C(g)\|_p \right) \right\|_q \\ &= \left\| \left(\|T_C(\ker f)\|_p \,, \|T_C(\mathrm{coker}\, f)\|_p \,, \|T_C(\ker g)\|_p \,, \|T_C(\mathrm{coker}\, g)\|_p \right) \right\|_q \\ &= c, \end{aligned}$$

where the second equality holds because the functor T_C preserves kernels and cokernels, and the last equality holds by Proposition 4.13.

To prove the other direction of the correspondence, we start from a span $T_C(X) \overset{\varphi}{\leftarrow} T_C(Z) \overset{\psi}{\rightarrow} T_C(Y)$ whose (p, q, D)-cost is

$$k := \left\| \left(\|\ker \varphi\|_p \,, \|\mathrm{coker}\, \varphi\|_p \,, \|\ker \psi\|_p \,, \|\mathrm{coker}\, \psi\|_p \right) \right\|_q ,$$

and we exhibit a span between X and Y whose (p, q, C)-cost equals k. Applying the functor T_ℓ, we obtain the span $X \xleftarrow{T_\ell(\varphi)} Z \xrightarrow{T_\ell(\psi)} Y$. To determine the (p, q, C)-cost of this span we observe that

$$\|\ker T_\ell(\varphi)\|_{p,C} = \|T_\ell(\ker \varphi)\|_{p,C} = \|T_C T_\ell(\ker \varphi)\|_p = \|\ker \varphi\|_p \,,$$

where the first equality holds because T_ℓ preserves kernels, the second equality is by Proposition 4.13, and the third equality holds because $T_C T_\ell = 1_{\mathsf{Tame}}$. Since similar equalities hold for coker $T_\ell(\varphi)$, ker $T_\ell(\psi)$, and coker $T_\ell(\psi)$, the (p, q, C)-cost of the span $X \xleftarrow{T_\ell(\varphi)} Z \xrightarrow{T_\ell(\psi)} Y$ equals k. □

Remark 4.20 Some of the pseudometrics between persistence modules that have been studied by other authors fall within the framework we have presented in this subsection and in Sect. 4.1. If C is a regular contour, the pseudometric denoted by d_C in (Chachólski and Riihimäki 2020, Sect. 6) coincide with our pseudometrics of the type $d^1_{\mathcal{S}^\infty, C}$. In particular, for the standard contour (Sect. 2.2) the pseudometric $d^1_{\mathcal{S}^\infty}$ coincides with the standard pseudometric already introduced in Scolamiero et al. (2017). As we already mentioned, the algebraic pseudometrics introduced in Skraba and Turner (2020, Sect. 7) are of the form $d^p_{\mathcal{S}^p}$, thus coinciding with our pseudometrics with the choice $p = q$ and for the standard contour. In Giunti et al. (2024), the authors propose a framework to study distances on abelian categories which is equivalent to noise systems on abelian categories. The authors of Bubenik et al. (2023) also study distances on abelian categories, introducing the notion of exact weight, which is more general than noise systems as the first axiom on short exact sequences is relaxed. The so-called path metric associated with an exact weight is defined for zigzags of morphisms of arbitrary finite length, but for the particular case of path metrics on noise systems considering spans is sufficient. In this case, the path metric coincides with a pseudometric of the form $d^1_{\mathcal{S}}$. In particular, the path metric $d_{\mu\circ\mathrm{dim}}$ between persistence modules studied in (Bubenik et al. 2023, Sect. 4) coincides with $d^1_{\mathcal{S}^1}$ in our notations, while the p-Wasserstein distances introduced by the authors are different from our pseudometrics $d^q_{\mathcal{S}^{p,C}}$.

4.4 Algebraic and combinatorial (p, C)-Wasserstein distances

In this subsection we consider Wasserstein distances between persistence diagrams. Here, we call these pseudometrics *combinatorial* Wasserstein distances, to distinguish them from the *algebraic* pseudometrics $d^q_{\mathcal{S}^{p,C}}$ defined on the class of persistence modules. We introduce a new family of combinatorial Wasserstein distances, parametrized by $p, q \in [1, \infty]$ and a regular action C, which generalize the Wasserstein distances commonly used in persistence theory. Finally, we prove isometry results involving the combinatorial Wasserstein distances and the algebraic Wasserstein distances $d^q_{\mathcal{S}^{p,C}}$ introduced in Sect. 4.2.

Let $U := \{(a, b) \in [0, \infty) \times [0, \infty] \mid a \leq b\}$ be a subset of the extended plane. A **persistence diagram** is a finite multiset $D = \{x_i\}_{i \in S}$ of elements of U. Since D is a multiset, it may happen that $x_i = x_k$ for some $i \neq k$. The **diagonal** Δ of $[0, \infty)$ is the set $\Delta := \{(a, a) \mid a \in [0, \infty)\} \subset U$. For all $p \in [1, \infty]$, we denote by d_p the metric on U induced by the p-norm, defined by $d_p(x, y) := \|x - y\|_p$ for all $x, y \in U$, and we denote $d_p(x, \Delta) := \inf_{z \in \Delta} d_p(x, z)$. As is easy to show, if $x = (a, b)$, then $d_p(x, \Delta) = d_p(x, \overline{z})$ with $\overline{z} := (\frac{a+b}{2}, \frac{a+b}{2})$.

Let $D = \{x_i\}_{i \in \{1,\ldots,m\}}$ and $D' = \{x'_j\}_{j \in \{1,\ldots,n\}}$ be persistence diagrams. For any $p, q \in [1, \infty]$, the (p, q)-**Wasserstein distance** between D and D' is defined by

$$W^q_p(D, D') := \inf_{\alpha} \left\| \left(\left\| (d_p(x_i, x'_{\alpha(i)}))_{i \in I} \right\|_q, \left\| (d_p(x_i, \Delta))_{i \in \{1,\ldots,m\} \setminus I} \right\|_q, \right.\right.$$
$$\left.\left. \left\| (d_p(\Delta, x'_j))_{j \in \{1,\ldots,n\} \setminus \alpha(I)} \right\|_q \right) \right\|_q,$$

where the infimum is over all injective functions $\alpha \colon I \to \{1, \ldots, n\}$, with $I \subseteq \{1, \ldots, m\}$.

Remark 4.21 We note that in the literature, the letters p and q are sometimes interchanged with respect to our notation of the parameters of Wasserstein distances between persistence diagrams. This is the case for instance in (Skraba and Turner 2020, Def. 2.7). Our choice of notation is motivated by symmetry with the definition of algebraic Wasserstein distances, where a norm $\|\cdot\|_q$ is used to "aggregate" costs expressed with respect to a norm $\|\cdot\|_p$.

Let $\mathcal{D}$ denote the set of all persistence diagrams. We define the class function $\mathrm{Dgm} \colon \mathsf{Tame} \to \mathcal{D}$ sending any persistence module X to the persistence diagram $\mathrm{Dgm}(X)$ such that $X \cong \bigoplus_{(a,b) \in \mathrm{Dgm}(X)} K(a, b)$, where we note that in the right-hand term each bar $K(a, b)$ appears the same number of times as the multiplicity of (a, b) in the multiset $\mathrm{Dgm}(X)$. By virtue of the barcode decomposition theorem (Theorem 2.2), the function $\mathrm{Dgm} \colon \mathsf{Tame} \to \mathcal{D}$ induces a bijection between the set $\mathsf{Tame}/_\sim$ of isomorphism classes of persistence modules and $\mathcal{D}$.

As proven in Skraba and Turner (2020), if $p = q$ then the algebraic distance $d^q_{\mathcal{S}^p}$ between persistence modules coincides with the combinatorial distance W^q_p between the associated persistence diagrams.

Theorem 4.22 (Skraba and Turner 2020) *For any $p \in [1, \infty]$ and for any persistence modules X and Y we have*

$$d^p_{\mathcal{S}^p}(X, Y) = W^p_p(\mathrm{Dgm}(X), \mathrm{Dgm}(Y)).$$

It is worth observing that the equality of Theorem 4.22 does not hold when $p \neq q$. For example, we can consider the persistence modules

$$X = K(a_1, a_1 + \ell_1) \oplus K(a_2, a_2 + \ell_2) \oplus K(a_3, a_3 + \ell_3)$$

with ℓ_1, ℓ_2, ℓ_3 positive real numbers, and 0, the zero module. Then, assuming $q < \infty$,

$$d^q_{\mathcal{S}^p}(X, 0) = \left(\left\| \left(\frac{\ell_1}{2}, \frac{\ell_2}{2}, \frac{\ell_3}{2} \right) \right\|^q_p + \left\| \left(\frac{\ell_1}{2}, \frac{\ell_2}{2}, \frac{\ell_3}{2} \right) \right\|^q_p \right)^{\frac{1}{q}}$$

(as we will prove in Lemma 4.23), while

$$W^q_p(\mathrm{Dgm}(X), \mathrm{Dgm}(0)) = \left(\left\| \left(\frac{\ell_1}{2}, \frac{\ell_1}{2} \right) \right\|^q_p + \left\| \left(\frac{\ell_2}{2}, \frac{\ell_2}{2} \right) \right\|^q_p + \left\| \left(\frac{\ell_3}{2}, \frac{\ell_3}{2} \right) \right\|^q_p \right)^{\frac{1}{q}}.$$

Given a regular contour C, we now define a function $\tau_C : U \to U$ as follows: for $x = (a, b) \in U$, we set $\tau_C(x) = (\ell(0, a), \ell(0, b))$, where $\ell(0, -)$ is the lifetime function associated with C (Sect. 2.2). If D is a persistence diagram, then by applying τ_C to each element of D we obtain a persistence diagram that we denote by $\tau_C(D)$. Hence, we have a function $\mathcal{D} \to \mathcal{D}$ which we denote again by τ_C, with a slight abuse of notation. If C is the standard contour, then τ_C is the identity function and in particular $\tau_C(D) = D$. Figure 1 illustrates a persistence diagram transformed by applying τ_C for a contour C of distance type.

Given a regular contour C, we define the **combinatorial** (p, C)**-Wasserstein distance** $W^p_{p,C}$ pulling back the pseudometric W^p_p via $\tau_C : \mathcal{D} \to \mathcal{D}$. Explicitly, for all persistence diagrams D and D', we define $W^p_{p,C}(D, D') := W^p_p(\tau_C(D), \tau_C(D'))$. If C is a regular action, then as a consequence of Corollary 4.9 we have $\mathrm{Dgm}(T_C(X)) = \tau_C(\mathrm{Dgm}(X))$, for every persistence module X. This implies, by virtue of Proposition 4.19 and Theorem 4.22, that

$$d^p_{\mathcal{S}^{p,C}}(X, Y) = W^p_{p,C}(\mathrm{Dgm}(X), \mathrm{Dgm}(Y)),$$

for all persistence modules X and Y.

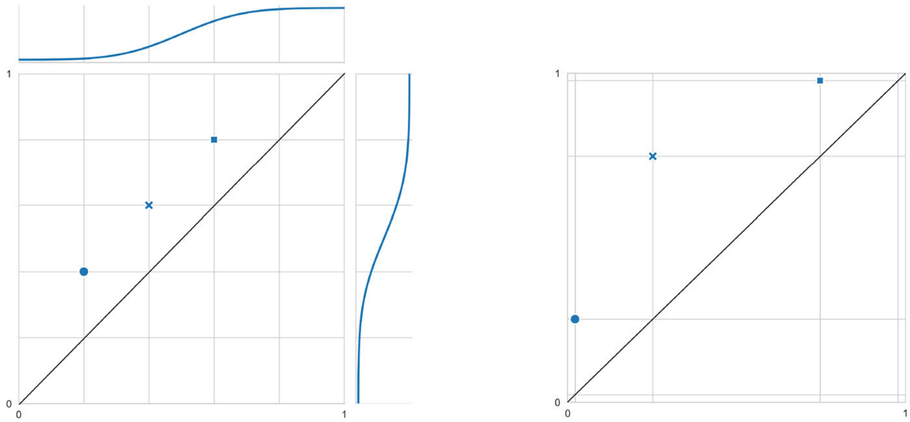


Fig. 1 **Left:** A persistence diagram $D = \{(0.2, 0.4), (0.4, 0.6), (0.6, 0.8)\}$. A contour C of distance type parametrized by a Gaussian density ($\mu = 0.5, \sigma = 0.15$) is chosen and the corresponding function $f(x) = \ell(0, x)$ (i.e. the Gaussian cumulative distribution function) is shown above and to the right of the persistence diagram. **Right:** The transformed persistence diagram $\tau_C(D) = \{(\ell(0, 0.2), \ell(0, 0.4)), (\ell(0, 0.4), \ell(0, 0.6)), (\ell(0, 0.6), \ell(0, 0.8))\}$. The regular grid from the left diagram has also been transformed to illustrate how τ_C stretches the plane

To summarize, for any $p \in [1, \infty]$ and any regular action C, we have a commutative diagram of isometries

$$\begin{tikzcd}
(\mathsf{Tame}, d^p_{\mathcal{S}^p,C}) \arrow[r, "\mathrm{Dgm}"] \arrow[d, "\tau_C"'] & (\mathcal{D}, W^p_{p,C}) \arrow[d, "\tau_C"] \\
(\mathsf{Tame}, d^p_{\mathcal{S}^p}) \arrow[r, "\mathrm{Dgm}"] & (\mathcal{D}, W^p_p)
\end{tikzcd}$$

4.5 Algebraic parametrized Wasserstein distances

The equivalence between algebraic and combinatorial Wasserstein distances for the case $p = q$, described in Sect. 4.4 or in Skraba and Turner (2020) for the standard contour, implies that in general Wasserstein distances cannot be expressed by an explicit (i.e., not involving an optimization problem) formula depending on the barcode decompositions of the persistence modules we are comparing. However for some special classes of persistence modules this is the case. The focus of this section is to present such formulas for the exact computation of algebraic Wasserstein distances. To avoid distinguishing the cases $q < \infty$ and $q = \infty$ in stating the results of this subsection, for $q = \infty$ we set by convention $\frac{1}{q} = 0$ and $2^{\frac{1-q}{q}} = 2^{-1}$.

Lemma 4.23 *For all persistence modules X and all $p, q \in [1, \infty]$ we have*

$$d^q_{\mathcal{S}^p}(X, 0) = 2^{\frac{1-q}{q}} \, \|X\|_p \, .$$

Proof Let $X = \bigoplus_{i=1}^{k} K(a_i, b_i)$ be a barcode decomposition of X, consider a persistence module of the form $Z = \bigoplus_{i=1}^{k} K(\frac{a_i+b_i}{2}, b_i)$ and a bar-to-bar morphism $f = \bigoplus_{i=1}^{k} f_i : Z \to X$, with each $f_i : K(\frac{a_i+b_i}{2}, b_i) \to K(a_i, b_i)$ a monomorphism between bars. Denote by $g = \bigoplus_{i=1}^{k} g_i : Z \to 0$ the zero map. The existence of the span $X \overset{f}{\hookleftarrow} Z \overset{g}{\to} 0$ implies that X and 0 are $2^{\frac{1-q}{q}} \|X\|_p$ close in q-norm (Definition 4.3), proving that $d^q_{\mathcal{S}^p}(X, 0) \leq 2^{\frac{1-q}{q}} \|X\|_p$. Indeed $\ker f = \operatorname{coker} g = 0$ and $\|\operatorname{coker} f\|_p = \|\ker g\|_p = \left\|(\frac{b_i-a_i}{2})_{i\in\{1,\ldots,k\}}\right\|_p$. The bound is obtained by computing $\left\|\left(\left\|(\frac{b_i-a_i}{2})_{i\in\{1,\ldots,k\}}\right\|_p, \left\|(\frac{b_i-a_i}{2})_{i\in\{1,\ldots,k\}}\right\|_p\right)\right\|_q = 2^{\frac{1-q}{q}} \|X\|_p$.

To prove the converse inequality, let us show that if $d^q_{\mathcal{S}^p}(X, 0) < \varepsilon$ then $2^{\frac{1-q}{q}} \|X\|_p < \varepsilon$. If $d^q_{\mathcal{S}^p}(X, 0) < \varepsilon$, then there exists a $(\varepsilon_1, \varepsilon_2, \varepsilon_3, 0)$-span $X \overset{\varphi}{\leftarrow} Z \twoheadrightarrow 0$ for some $\varepsilon_1, \varepsilon_2, \varepsilon_3$ in $[0, \infty)$ such that $\|(\varepsilon_1, \varepsilon_2, \varepsilon_3)\|_q < \varepsilon$. Note that $X \hookleftarrow \operatorname{im}\varphi \twoheadrightarrow 0$ is then a $(0, \varepsilon_2, \varepsilon_3, 0)$-span. Consider the short exact sequence $\operatorname{im}\varphi \hookrightarrow X \twoheadrightarrow \operatorname{coker}\varphi$. Since $\operatorname{coker}\varphi \in \mathcal{S}^p_{\varepsilon_2}$ and $\operatorname{im}\varphi \in \mathcal{S}^p_{\varepsilon_3}$, by the third axiom of noise systems we get $X \in \mathcal{S}^p_{\varepsilon_2+\varepsilon_3}$, and so we get $\|X\|_p \leq \varepsilon_2 + \varepsilon_3$ by definition of $\mathcal{S}^p$. Furthermore, by inequalities (1) between p-norms on $\mathbb{R}^2$, $\varepsilon_2 + \varepsilon_3 = \|(\varepsilon_2, \varepsilon_3)\|_1 \leq 2^{1-\frac{1}{q}} \|(\varepsilon_2, \varepsilon_3)\|_q < 2^{1-\frac{1}{q}} \varepsilon$. Therefore we have $\|X\|_p < 2^{1-\frac{1}{q}} \varepsilon$ or equivalently $2^{\frac{1-q}{q}} \|X\|_p < \varepsilon$. We conclude that $d^q_{\mathcal{S}^p}(X, 0) \geq 2^{\frac{1-q}{q}} \|X\|_p$, and therefore $d^q_{\mathcal{S}^p}(X, 0) = 2^{\frac{1-q}{q}} \|X\|_p$. □

Remark 4.24 The formula $d^q_{\mathcal{S}^p}(X, 0) = 2^{\frac{1-q}{q}} \|X\|_p$ of Lemma 4.23 was already shown for the case $p = q$ in Skraba and Turner (2020) by using the correspondence between combinatorial and algebraic Wasserstein distances.

The proof of Lemma 4.23 can be easily extended to the case of a regular action C. In this case, we have

$$d^q_{\mathcal{S}^p,C}(X, 0) = d^q_{\mathcal{S}^p}(T_C(X), 0) = 2^{\frac{1-q}{q}} \|T_C(X)\|_p = 2^{\frac{1-q}{q}} \|X\|_{p,C}\,, \tag{6}$$

where the first equality holds by Proposition 4.19, the second by Lemma 4.23 and the third by Proposition 4.13. Similar arguments can be applied to all the results of this subsection. For exposition purposes we consider the case of the standard contour throughout the section and collect generalizations of the main results at the end of the subsection in Proposition 4.32.

Proposition 4.25 *Let X, Y, V be persistence modules. Then, for every $p, q \in [1, \infty]$,*

$$d^q_{\mathcal{S}^p}(X \oplus V, Y \oplus V) \leq d^q_{\mathcal{S}^p}(X, Y).$$

Proof It suffices to observe that for any span $X \overset{f}{\leftarrow} Z \overset{g}{\to} Y$, the span $X \oplus V \xleftarrow{f\oplus 1} Z \oplus V \xrightarrow{g\oplus 1} Y \oplus V$ has the same cost. □

Remark 4.26 Note that by considering $Y = 0$, Proposition 4.25 gives

$$d^q_{\mathcal{S}^p}(X \oplus V, V) \leq d^q_{\mathcal{S}^p}(X, 0) = 2^{\frac{1-q}{q}} \|X\|_p$$

The converse inequality $d^q_{\mathcal{S}^p}(X \oplus V, V) \geq d^q_{\mathcal{S}^p}(X, 0) = 2^{\frac{1-q}{q}} \|X\|_p$ does not hold in general, as illustrated in the following example. Consider $p = q = 2$, $X = K(0, 6)$ and $V = K(1, 5) \oplus K(2, 4)$. By Lemma 4.23 we have that $d^q_{\mathcal{S}^p}(X, 0) = \frac{1}{\sqrt{2}} \cdot 6 = \sqrt{18}$. However, $X \oplus V$ and $Y \oplus V$ are $\sqrt{6}$-close via the following $(0, \sqrt{3}, \sqrt{3}, 0)$-span

$$\begin{aligned} K(0, 6) \oplus K(1, 5) \oplus K(2, 4) &\xleftarrow{f_1 \oplus f_2 \oplus f_3} K(1, 6) \oplus K(2, 5) \\ \oplus\, K(3, 4) &\xrightarrow{g_1 \oplus g_2 \oplus g_3} K(1, 5) \oplus K(2, 4) \oplus 0 \end{aligned}$$

implying that $d^q_{\mathcal{S}^p}(X \oplus V, Y \oplus V) \leq \sqrt{6} < \sqrt{18} = d^q_{\mathcal{S}^p}(X, Y)$. This example is based on the fact that given a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ realizing the distance between X and Y, the span $X \oplus V \xleftarrow{f \oplus 1} Z \oplus V \xrightarrow{g \oplus 1} Y \oplus V$ not always is the one achieving the distance between $X \oplus V$ and $Y \oplus V$.

Let $\{K(a_i, b_i)\}_{i \in \{1,\ldots,k\}}$ be a sequence of bars ordered non-decreasingly by length, that is, $b_1 - a_1 \leq b_2 - a_2 \leq \cdots \leq b_k - a_k$. For $j \in \{1, \ldots, k\}$, consider $Z := \bigoplus_{i=1}^{j} K(a_i, b_i)$ and $Y := \bigoplus_{i=j+1}^{k} K(a_i, b_i)$. The remainder of this section is devoted to proving that, in this case, $d^q_{\mathcal{S}^p}(Y \oplus Z, Y) = d^q_{\mathcal{S}^p}(Z, 0) = 2^{\frac{1-q}{q}} \|Z\|_p$. In Sect. 5, this result will be used for the computation of the stable rank of a persistence module with respect to $d^q_{\mathcal{S}^p}$.

Proposition 4.27 *Let $\mathcal{S}$ be a noise system. For any $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$-span $X \leftarrow Z \rightarrow Y$ of persistence modules there is a mono-epi $(0, \varepsilon_2, \varepsilon_3, 0)$-span $X \hookleftarrow \operatorname{im} f \twoheadrightarrow P$ such that* $\operatorname{rank}(P) \leq \operatorname{rank}(Y)$.

Proof By Theorem 3.14 and Remark 3.2, if $U \hookrightarrow V$ is a monomorphism between persistence modules, then $\operatorname{rank}(U) \leq \operatorname{rank}(V)$, and similarly if $V \twoheadrightarrow U$ is an epimorphism, then $\operatorname{rank}(U) \leq \operatorname{rank}(V)$. Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$-span of persistence modules, and consider the following diagram in Tame, where the square is a push-out:

$$\begin{tikzcd} & & Z \arrow[dl, two heads, "f"'] \arrow[dr, two heads, "g"] & & \\ & \operatorname{im} f \arrow[dl, hook', "j"'] \arrow[dr, two heads, "g'"] & & \operatorname{im} g \arrow[dl, two heads, "f'"] \arrow[dr, hook, "i"] & \\ X & & P & & Y \end{tikzcd}$$

Since f' is an epimorphism and i is a monomorphism, $\operatorname{rank}(P) \leq \operatorname{rank}(\operatorname{im} g) \leq \operatorname{rank}(Y)$. We consider the span $X \overset{j}{\hookleftarrow} \operatorname{im} f \overset{g'}{\twoheadrightarrow} P$. Clearly, the kernel of the corestriction $g \colon Z \twoheadrightarrow \operatorname{im} g$ still belongs to $\mathcal{S}_{\varepsilon_3}$, and its cokernel is zero. Then, by Proposition 8.1

in Scolamiero et al. (2017), $\ker g' \in \mathcal{S}_{\varepsilon_3}$ and $\operatorname{coker} g' = 0$. The kernel of j is 0, while its cokernel belongs to $\mathcal{S}_{\varepsilon_2}$, as it coincides with the cokernel of $f : Z \to X$. $\square$

Lemma 4.28 *Let $p, q \in [1, \infty]$, and let $[a_i, b_i]$ be nonempty intervals in $[0, \infty)$, for $i \in \{1, \ldots, j\}$. The function $\gamma : \prod_{i=1}^{j} [a_i, b_i] \to [0, \infty)$ defined by*

$$\gamma(x_1, \ldots, x_j) := \Big\| \Big(\big\|(x_1 - a_1, \ldots, x_j - a_j)\big\|_p , \big\|(b_1 - x_1, \ldots, b_j - x_j)\big\|_p \Big) \Big\|_q$$

has a global minimum at $(\frac{a_1+b_1}{2}, \ldots, \frac{a_j+b_j}{2})$.

Proof The function γ is continuous with a compact domain, so it admits a global minimum by the extreme value theorem. Moreover, it is convex because norms are convex functions.

Write $a = (a_1, \ldots, a_j)$, $b = (b_1, \ldots, b_j)$ and $x = (x_1, \ldots, x_j)$ in $\mathbb{R}^j$. Since $\gamma(x) = \gamma(a + b - x)$ for every x, the function γ is invariant under point reflection through $\frac{a+b}{2}$. By convexity, we conclude that $\frac{a+b}{2}$ is a global minimum of γ. $\square$

Proposition 4.29 *Let $X = \bigoplus_{i=1}^{k} K(a_i, b_i)$, with the bars ordered non-decreasingly by length. Let $j \in \{1, \ldots, k\}$, and let $p, q \in [1, \infty]$. Then, any persistence module Y with $\operatorname{rank}(Y) \le \operatorname{rank}(X) - j$ is such that*

$$d^q_{\mathcal{S}^p}(X, Y) \ge 2^{\frac{1-q}{q}} \Big\| \bigoplus_{i=1}^{j} K(a_i, b_i) \Big\|_p .$$

Proof We prove the claim by contradiction. Suppose that there exists a persistence module Y such that $\operatorname{rank}(Y) \le \operatorname{rank}(X) - j$ and

$$d^q_{\mathcal{S}^p}(X, Y) < 2^{\frac{1-q}{q}} \Big\| \bigoplus_{i=1}^{j} K(a_i, b_i) \Big\|_p .$$

By definition, there exists a span $X \xleftarrow{f} Z \xrightarrow{g} Y$ such that

$$\big\| \big(\|\ker f\|_p , \|\operatorname{coker} f\|_p , \|\ker g\|_p , \|\operatorname{coker} g\|_p \big) \big\|_q < 2^{\frac{1-q}{q}} \Big\| \bigoplus_{i=1}^{j} K(a_i, b_i) \Big\|_p . \tag{7}$$

By Proposition 4.27 we can assume (possibly after replacing Y with a persistence module of smaller or equal rank) that the span above is mono-epi, that is, of the form $X \overset{f}{\hookleftarrow} Z \overset{g}{\twoheadrightarrow} Y$. By Theorems 3.14 and 3.16, we can moreover assume that f and g are bar-to-bar morphisms.

Thus, we can consider a barcode decomposition $Z = \bigoplus_{i=1}^{k} Z_i$, with some of the Z_i possibly zero, and a barcode decomposition $Y = \bigoplus_{i=1}^{k} Y_i$, with at least j of the Y_i equal to zero by assumption, together with morphisms between bars $K(a_i, b_i) \overset{f_i}{\hookleftarrow} Z_i \overset{g_i}{\twoheadrightarrow} Y_i$ such that $f = \bigoplus_{i=1}^{k} f_i$ and $g = \bigoplus_{i=1}^{k} g_i$. Let $I \subseteq \{1, \ldots, k\}$, with $|I| \ge j$, be the subset of the indices i such that $Y_i = 0$. For every $i \in I$, we have

$K(a_i, b_i) \overset{f_i}{\hookleftarrow} Z_i \overset{g_i}{\twoheadrightarrow} 0$, with $Z_i = K(x_i, b_i)$ for some $a_i \le x_i \le b_i$, where $K(b_i, b_i)$ denotes the zero module. Since $\ker f = \bigoplus_{i=1}^k \ker f_i$ and $\operatorname{coker} f = \bigoplus_{i=1}^k \operatorname{coker} f_i$, by Remark 3.2 we observe that $\bigoplus_{i\in I} K(a_i, x_i)$ is a direct summand of $\operatorname{coker} f$, and similarly that $\bigoplus_{i\in I} K(x_i, b_i)$ is a direct summand of $\ker g$, which gives

$$\begin{aligned}\|\operatorname{coker} f\|_p &\ge \|\oplus_{i\in I} K(a_i, x_i)\|_p = \|(x_i - a_i)_{i\in I}\|_p\,,\\ \|\ker g\|_p &\ge \|\oplus_{i\in I} K(x_i, b_i)\|_p = \|(b_i - x_i)_{i\in I}\|_p\,.\end{aligned}$$

If $b_i < \infty$ for all $i \in I$, it is easy to show using Lemma 4.28 that the cost of the span is

$$\left\|\left(\|\operatorname{coker} f\|_p\,, \|\ker g\|_p\right)\right\|_q \ge 2^{\frac{1-q}{q}} \|(b_i - a_i)_{i\in I}\|_p = 2^{\frac{1-q}{q}} \left\|\bigoplus_{i\in I} K(a_i, b_i)\right\|_p,$$

and the same inequality clearly holds if $b_i = \infty$ for some $i \in I$. However, since $|I| \ge j$, the right-hand side of the inequality cannot be smaller than

$$2^{\frac{1-q}{q}} \left\|(b_i - a_i)_{i\in\{1,\ldots,j\}}\right\|_p = 2^{\frac{1-q}{q}} \left\|\bigoplus_{i=1}^j K(a_i, b_i)\right\|_p,$$

and this contradicts (7). □

Proposition 4.30 *Let $X = \bigoplus_{i=1}^k K(a_i, b_i)$, with the bars ordered non-decreasingly by length. Let $j \in \{1, \ldots, k\}$, and let $Y = \bigoplus_{i=j+1}^k K(a_i, b_i)$ (with $Y = 0$ when $j = k$). Then, for all $p, q \in [1, \infty]$,*

$$d^q_{\mathcal{S}^p}(X, Y) = 2^{\frac{1-q}{q}} \left\|\bigoplus_{i=1}^j K(a_i, b_i)\right\|_p. \tag{8}$$

Proof Since $\operatorname{rank}(Y) = \operatorname{rank}(X) - j$, Proposition 4.29 gives us the inequality

$$d^q_{\mathcal{S}^p}(X, Y) \ge 2^{\frac{1-q}{q}} \left\|\bigoplus_{i=1}^j K(a_i, b_i)\right\|_p.$$

The other inequality, as already noticed in Remark 4.26, follows from Proposition 4.25 and Lemma 4.23 showing that $d^q_{\mathcal{S}^p}(Z \oplus Y, Y) \le d^q_{\mathcal{S}^p}(Z, 0) = 2^{\frac{1-q}{q}} \|Z\|_p$ with $Z = \bigoplus_{i=1}^j K(a_i, b_i)$. □

In the final part of this subsection we generalize some results from the case of the standard contour to the case of any regular action C.

Definition 4.31 Let C be a regular contour, and let $X = \bigoplus_{i=1}^k K(a_i, b_i)$. We say that (the barcode decomposition of) X has **bars ordered non-decreasingly by lifetime** if $\ell(a_1, b_1) \le \ell(a_2, b_2) \le \cdots \le \ell(a_k, b_k)$, where ℓ denotes the lifetime function associated with C (see Sect. 2.2).

Proposition 4.32 *Let C be a regular action, and let $p, q \in [1, \infty]$. Let $X = \bigoplus_{i=1}^{k} K(a_i, b_i)$, with bars ordered non-decreasingly by lifetime, and let $j \in \{1, \ldots, k\}$. Then, for all persistence modules Y,*

1. *if* $\operatorname{rank}(Y) \leq \operatorname{rank}(X) - j$, *then*

$$d^q_{\mathcal{S}^{p,C}}(X, Y) \geq 2^{\frac{1-q}{q}} \left\| \bigoplus_{i=1}^{j} K(a_i, b_i) \right\|_{p,C};$$

2. *if $Y = \bigoplus_{i=j+1}^{k} K(a_i, b_i)$ (with the convention $Y = 0$ when $j = k$), then*

$$d^q_{\mathcal{S}^{p,C}}(X, Y) = 2^{\frac{1-q}{q}} \left\| \bigoplus_{i=1}^{j} K(a_i, b_i) \right\|_{p,C}.$$

Proof The first statement follows from

$$\begin{aligned} d^q_{\mathcal{S}^{p,C}}(X, Y) &= d^q_{\mathcal{S}^p}(T_C(X), T_C(Y)) \\ &\geq 2^{\frac{1-q}{q}} \left\| \bigoplus_{i=1}^{j} T_C(K(a_i, b_i)) \right\|_p \\ &= 2^{\frac{1-q}{q}} \left\| \bigoplus_{i=1}^{j} K(a_i, b_i) \right\|_{p,C} \end{aligned}$$

where we are using in sequence Proposition 4.19, Proposition 4.29 (observing that the length of a bar $T_C(K(a, b))$ coincides with the lifetime $\ell(a, b)$ of $K(a, b)$, see Proposition 4.8), and Proposition 4.13. The second statement is proven similarly, using Proposition 4.30. □

5 Wasserstein stable ranks: computations and stability

In Sect. 4 it was shown that the Wasserstein distances $d^q_{\mathcal{S}^{p,C}}$ are pseudometrics on Tame. They can therefore be used in the framework of hierarchical stabilization (see Sect. 2.5) to produce stable invariants of persistence modules. The focus of this section is on one type of such invariants, the **Wasserstein stable ranks**, which are the hierarchical stabilization of the rank function with respect to Wasserstein distances $d^q_{\mathcal{S}^{p,C}}$. Denoting $d^q_{\mathcal{S}^{p,C}}$ by d, the stability result for stable ranks (Proposition 2.5) states that for every pair of persistence modules X and Y

$$d(X, Y) \geq d_{\bowtie}\left(\widehat{\operatorname{rank}}_d(X), \widehat{\operatorname{rank}}_d(Y)\right).$$

In the case where $p = q$ and C is the standard contour, combining the above inequality with the stability results of Skraba and Turner (2020) gives several stability results of Wasserstein stable ranks with respect to perturbation of the original data. In particular, (Skraba and Turner 2020, Theorem 4.8) expresses stability with respect to sublevel set filtrations of monotone functions on cellular complexes, (Skraba and Turner 2020, Theorem 5.1) expresses stability with respect to the construction of cubical complexes from grey scale images, and (Skraba and Turner 2020, Theorem 5.9), expresses stability with respect to Wasserstein distance between point clouds when using the Vietoris-Rips construction.

In order to use the Wasserstein stable ranks in applications, it is important to be able to efficiently compute them as well as distances between them. In this section we use computations of Wasserstein distances from Sect. 4 to derive a formula for the Wasserstein stable rank and propose a convenient formulation of the interleaving distance between stable ranks.

Having defined a rich family of Wasserstein distances $d^q_{\mathcal{S}p,C}$, it is natural to ask whether we can in a supervised learning context search for an optimal distance for a problem at hand. Choosing a suitable parametrization of a contour and leveraging the simple expression of the interleaving distance between Wasserstein stable ranks, in Sect. 5.3 we set up a simple metric learning problem with the aim of observing the interaction between the parameter p and the parameters related to the contour C within the learning. Preliminary results on the optimization of only a contour in a metric learning framework are presented in Gäfvert (2018).

5.1 Computation of the stable rank with Wasserstein distances

The results of this subsection provide explicit formulas to compute the stable rank with respect to the Wasserstein distances $d^q_{\mathcal{S}p,C}$ introduced in Sect. 4. We begin by considering the case $p<\infty$. As in the previous section, if $q=\infty$ we set by convention $\frac{1}{q}=0$ and $2^{\frac{1-q}{q}}=2^{-1}$.

Proposition 5.1 *Let $p\in[1,\infty)$ and $q\in[1,\infty]$, let C be a regular action, and let d denote the pseudometric $d^q_{\mathcal{S}p,C}$. Let $X=\bigoplus_{i=1}^k K(a_i,b_i)$, with bars ordered non-decreasingly by lifetime (Definition 4.31), and let $n:=|\{i\in\{1,\dots,k\}\mid b_i<\infty\}|$ denote the number of finite bars of X. Then, there exist real numbers $0=t_0<t_1<t_2<\cdots<t_n$ such that the stable rank function $\widehat{\operatorname{rank}}_d(X)\colon[0,\infty)\to[0,\infty)$ is constant on the intervals $[t_0,t_1)$, $[t_1,t_2)$,..., $[t_{n-1},t_n)$, $[t_n,\infty)$, and*

$$\widehat{\operatorname{rank}}_d(X)(t_j)=\operatorname{rank}(X)-j,$$

for every $j\in\{0,1,\dots,n\}$. Furthermore,

$$t_j=2^{\frac{1-q}{q}}\left\|\bigoplus_{i=1}^j K(a_i,b_i)\right\|_{p,C}=2^{\frac{1-q}{q}}\left\|(\ell(a_1,b_1),\dots,\ell(a_j,b_j))\right\|_p$$

for every $j\in\{1,\dots,n\}$, where ℓ is the lifetime function associated with C.

Proof For every $j\in\{1,\dots,k\}$, by Proposition 4.32 $Y_j:=\bigoplus_{i=j+1}^k K(a_i,b_i)$ is the closest persistence module to X (in the pseudometric $d^q_{\mathcal{S}p,C}$) such that $\operatorname{rank}(Y_j)=\operatorname{rank}(X)-j$. We have

$$d^q_{\mathcal{S}p,C}(X,Y_j)=2^{\frac{1-q}{q}}\left\|\bigoplus_{i=1}^j K(a_i,b_i)\right\|_{p,C}=:t_j,$$

with $t_j < \infty$ if, and only if, $j \in \{1, \ldots, n\}$. Lastly, we observe that $0 = t_0 < t_1 < t_2 < \cdots < t_n$ as a consequence of the assumption $p < \infty$. □

In particular, when $p < \infty$, the value of the piecewise constant function $\widehat{\mathrm{rank}}_d(X)$ can only decrease by 1 at every discontinuity point t_j. For $p = \infty$, the stable rank has a slightly different behavior. Even though we can define the sequence of real numbers $(t_j)_j$ as in Proposition 5.1, we only have $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ instead of strict inequalities. Letting s_m denote the m^{th} smallest value in $\{t_j\}_j$ we obtain a sequence $0 = s_0 < s_1 < s_2 < \cdots < s_{n'}$ such that the stable rank with respect to the pseudometric $d := d^q_{S^{\infty},C}$ is constant on the intervals $[s_0, s_1)$,…, $[s_{n'}, \infty)$, taking the values

$$\widehat{\mathrm{rank}}_d(X)(s_m) = \mathrm{rank}(X) - \max\{j \mid t_j = s_m\}.$$

An explicit formula for the stable rank in the case $p = \infty$ and $q = 1$ was first given in Chachólski and Riihimäki (2020).

Remark 5.2 We observe that for a persistence module X of rank k, once the k bars in the barcode decomposition of X have been ordered non-decreasingly by lifetime, the complexity of computing the discontinuity points of the the Wasserstein stable rank using Proposition 5.1 is linear in k. Therefore the computational complexity of the Wasserstein stable rank is $O(k \log k)$, determined by the complexity of the sorting algorithm to order the bars non-decreasingly by lifetime.

5.2 Interleaving distance between stable ranks

The aim of this subsection is to propose a convenient expression for the interleaving distance (Sect. 2.5) between two non-increasing piecewise constant functions. We assume functions to take only finitely many values, that is the case of stable ranks which will be the object of our study. Let $f, g \colon [0, \infty) \to [0, \infty)$ be non-increasing piecewise constant functions. If $\lim_{t\to\infty} f(t) \neq \lim_{t\to\infty} g(t)$, then $d_{\bowtie}(f, g) = \infty$. For the computation of the interleaving distance we can therefore assume that the functions f and g have the same limit value and denote it by L. Given a non-increasing piecewise constant function $f \colon [0, \infty) \to [0, \infty)$ with limit value L, we define the non-increasing piecewise constant function $f^{-1} \colon [L, \infty) \to [0, \infty)$ with values $f^{-1}(y) := \inf\{t \mid f(t) \leq y\}$. If in addition the function f is right-continuous, then $f^{-1}(y) = \min\{t \mid f(t) \leq y\}$. We observe that for every right-continuous non-increasing piecewise constant function f we have $f^{-1}(f(t)) \leq t$ for all t, and equality holds if t is a discontinuity point of f. Moreover, $f(f^{-1}(y)) \leq y$ for all $y \geq L$, and equality holds if $y \in \mathrm{im}\, f$. Our focus in this subsection will be on the discontinuity points $\{t_i\}$ of f and on the values in $\mathrm{im}\, f$, rather than on the full domain and codomain of f, thus justifying our use of the notation f^{-1}.

Proposition 5.3 *Consider two right-continuous non-increasing piecewise constant functions $f, g \colon [0, \infty) \to [0, \infty)$ having the same limit value L. Using the nota-*

tion introduced above, we have:

$$d_{\bowtie}(f, g) = \|f^{-1} - g^{-1}\|_{\infty}.$$

Proof Let us define the following subset of $[0, \infty)$,

$$A(f, g) := \{\varepsilon \in [0, \infty) \mid f(t) \geq g(t + \varepsilon) \text{ and } g(t) \geq f(t + \varepsilon), \text{ for all } t \in [0, \infty)\}.$$

Remember that, by definition, $d_{\bowtie}(f, g) = \inf A(f, g)$.

We first prove that $d_{\bowtie}(f, g) \geq \|f^{-1} - g^{-1}\|_{\infty}$. Let $\varepsilon \in A(f, g)$. Then, for all $y \geq L$, we have $y \geq f(f^{-1}(y)) \geq g(f^{-1}(y) + \varepsilon)$. Composing by the non-increasing function g^{-1} and recalling that $g^{-1}(g(t)) \leq t$ for all t, we obtain $f^{-1}(y)+\varepsilon \geq g^{-1}(y)$. We have thus shown that $g^{-1}(y) - f^{-1}(y) \leq \varepsilon$, for all $y \geq L$ and $\varepsilon \in A(f, g)$, which implies $g^{-1}(y) - f^{-1}(y) \leq d_{\bowtie}(f, g)$, for all $y \geq L$. By symmetry in the roles of f and g, we conclude that $|g^{-1}(y) - f^{-1}(y)| \leq d_{\bowtie}(f, g)$, for all $y \geq L$.

We now prove that $d_{\bowtie}(f, g) \leq \|f^{-1} - g^{-1}\|_{\infty}$ by showing that $\varepsilon := \|f^{-1} - g^{-1}\|_{\infty}$ is in $A(f, g)$. By the definition of ε, $g^{-1}(y) \leq f^{-1}(y) + \varepsilon$ for every $y \in [L, \infty)$. Moreover if $y = f(t)$ for $t \in [0, \infty)$, as discussed above, we have $f^{-1}(y) = f^{-1}(f(t)) \leq t$, which together with the fact that g is a non-increasing function proves the following inequalities:

$$f(t) = y \geq g(g^{-1}(y)) \geq g(f^{-1}(y) + \varepsilon) \geq g(t + \varepsilon).$$

By symmetry, we also get $g(t) \geq f(t + \varepsilon)$, and we conclude that $\varepsilon \in A(f, g)$. □

Let $X = \bigoplus_{i=1}^{k} K(a_i^x, b_i^x)$ be a persistence module with bars ordered non-decreasingly by lifetime, and with n finite bars. Let $f := \widehat{\text{rank}}_d(X)$ denote the corresponding Wasserstein stable rank with respect to the distance $d = d^q_{\mathcal{S}p,C}$, for some $p, q \in [1, \infty]$ and a regular action C. The sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ such that f is constant on the intervals $[t_0, t_1)$, $[t_1, t_2)$,..., $[t_{n-1}, t_n)$, $[t_n, \infty)$ defined in Sect. 5.1 is enough to encode f^{-1} as a finite vector $\hat{f}^{-1} := (\hat{f}_i^{-1})_{i \in \{0,\ldots,n\}}$, where

$$\hat{f}_i^{-1} := t_{n-i} = 2^{\frac{1-q}{q}} \left\| \left(\ell\left(a_1^x, b_1^x\right), \ldots, \ell\left(a_{n-i}^x, b_{n-i}^x\right)\right) \right\|_p$$

for $i \in \{0, \ldots, n-1\}$ and $\hat{f}_n^{-1} = 0$. Indeed, the limit value of f is the number $L = k - n$ of infinite bars of X; for all $i \in \{0, \ldots, n\}$ we have $f^{-1}(L + i) = \hat{f}_i^{-1}$, and for any $y \in [L, \infty)$ the value $f^{-1}(y)$ equals $f^{-1}(L + i)$, where $i \in \{0, \ldots, n\}$ is the largest integer such that $L + i \leq y$.

Let $Y = \bigoplus_{i=1}^{l} K(a_i^y, b_i^y)$ be another persistence module with bars ordered non-decreasingly by lifetime, and with m finite bars. Suppose that X and Y have the same number of infinite bars, $L = k - n = l - m$. Let $g := \widehat{\text{rank}}_d(Y)$ denote the Wasserstein stable rank of Y with respect to the distance $d = d^q_{\mathcal{S}p,C}$. The interleaving distance between f and g can then be written as the L^{∞} norm of the vector $(\hat{f}_i^{-1} -$

$\hat{g}_i^{-1})_{i\in\{0,\dots,\min(n,m)\}}$ with components

$$\hat{f}_i^{-1} - \hat{g}_i^{-1} = 2^{\frac{1-q}{q}}\left(\left\|\left(\ell - (a_1^x, b_1^x), \dots, \ell\left(a_{n-i}^x, b_{n-i}^x\right)\right)\right\|_p \right. \\ \left. - \left\|\left(\ell(a_1^y, b_1^y), \dots, \ell\left(a_{m-i}^y, b_{m-i}^y\right)\right)\right\|_p\right) \tag{9}$$

for $i \in \{0, \dots, \min(n, m) - 1\}$, and last component

$$\hat{f}_{\min\{n,m\}}^{-1} - \hat{g}_{\min\{n,m\}}^{-1} = \begin{cases} -\|(\ell(a_1^y, b_1^y), \dots, \ell(a_{m-n}^y, b_{m-n}^y))\|_p & \text{if } \min(n, m) = n < m \\ \|(\ell(a_1^x, b_1^x), \dots, \ell(a_{n-m}^x, b_{n-m}^x))\|_p & \text{if } \min(n, m) = m < n \\ 0 & \text{if } n = m. \end{cases}$$

We observe that the considered vector encodes the function $(f^{-1} - g^{-1})$ on the intervals $[L, L+1), \dots, [\min(k, l), \min(k, l)+1)$ of length one where both f^{-1} and g^{-1} are constant. Since f^{-1} and g^{-1} are nonincreasing and for $x \geq \min(k, l)$ either $f^{-1}(x) = 0$ or $g^{-1}(x) = 0$, it is enough to consider those intervals.

Remark 5.4 For two persistence modules X and Y both of rank k, the complexity of computing the interleaving distance is dominated by the sorting of the bars in the respective barcode decompositions of X and Y, since forming the vector as in (9) and computing its L^∞ norm can be done linearly in k. The computational complexity of the interleaving distance between Wasserstein stable ranks is thus $O(k \log k)$.

Example 5.5 Consider a persistence module $Y = \bigoplus_{i=1}^3 K(a_i, b_i)$ with bars ordered non-decreasingly by lifetime and $X = K(a_0, b_0) \oplus Y$ such that $\varepsilon := \ell(a_0, b_0) \leq \ell(a_1, b_1)$. By using the formula (9) and observing that

$$\|(\ell(a_0, b_0), \dots, \ell(a_i, b_i))\|_p - \|(\ell(a_1, b_1), \dots, \ell(a_i, b_i))\|_p \leq \ell(a_0, b_0)$$

for $i \in \{1, 2, 3\}$ by properties (1) and (2) of p-norms, we see that the interleaving distance between $\widehat{\text{rank}}_d(X)$ and $\widehat{\text{rank}}_d(Y)$ with $d = d^q_{\mathcal{S}p,C}$ is given by $2^{\frac{1-q}{q}}\varepsilon$. Note that by Proposition 4.32 we know $d^q_{\mathcal{S}p,C}(X, Y) = 2^{\frac{1-q}{q}}\|K(a_0, b_0)\|_{p,C} = 2^{\frac{1-q}{q}}\varepsilon$. Therefore in this case the interleaving distance between stable ranks with respect to Wasserstein distance coincides with the Wasserstein distance between X and Y. Note however that this is not always the case. The Wasserstein stable ranks of X and Y with respect to $d^q_{\mathcal{S}p,C}$, with parameters $q = 1$, $p = 2$ and C the standard contour, are shown in Fig. 2, together with their "inverse" functions which are used for the computation of the interleaving distance.

Let us keep denoting $d^q_{\mathcal{S}p,C}$ by d. It follows from triangle inequality and Lemma 4.23 that:

$$d(X, Y) \geq 2^{\frac{1-q}{q}}\,|\|X\|_p - \|Y\|_p|.$$

This inequality can be refined by

$$d(X, Y) \geq d_{\bowtie}\left(\widehat{\text{rank}}_d(X), \widehat{\text{rank}}_d(Y)\right) \geq 2^{\frac{1-q}{q}}\,|\|X\|_p - \|Y\|_p|,$$

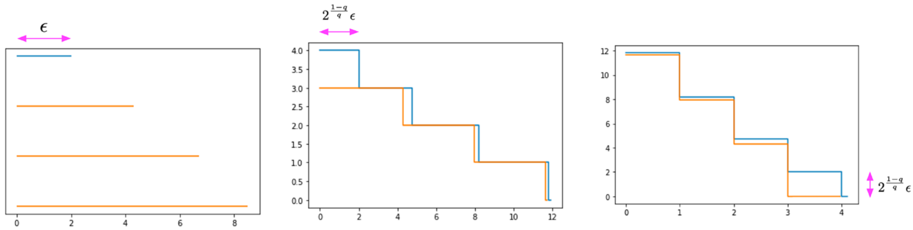


Fig. 2 Schematic representation of the computation of the interleaving distance in Example 5.5. **Left:** Barcode decomposition of Y in orange and bar $K(a_0, b_0)$ in blue. **Middle:** Stable ranks computed with standard contour, $q = 1$ and $p = 2$. The functions $\widehat{\text{rank}}_d(X)$ and $\widehat{\text{rank}}_d(Y)$ are represented in blue and orange, respectively. **Right:** Inverse stable ranks for the computation of interleaving distance, with $\widehat{\text{rank}}_d^{-1}(X)$ in blue and $\widehat{\text{rank}}_d^{-1}(Y)$ in orange. The interleaving distance between stable ranks can be computed as $\|\widehat{\text{rank}}_d^{-1}(X) - \widehat{\text{rank}}_d^{-1}(Y)\|_\infty = 2^{\frac{1-q}{q}}\varepsilon$, illustrated with the pink arrow (colour figure online)

where the first inequality is given by the stability theorem of hierarchical stabilization (Proposition 2.5) and the second inequality is provided by the characterization of interleaving distances between stable ranks in Proposition 5.3 and Eq. (9) stating that the interleaving distance between the Wasserstein stable ranks of persistence modules X and Y is the L^∞ norm of a vector with 0^{th} component $2^{\frac{1-q}{q}}|\|X\|_p - \|Y\|_p|$.

An example where the second inequality is strict is provided by Example 5.5 for $p > 1$, while an example where this is an equality is provided in the case $Y = 0$ by Lemma 4.23. A simple example in which the first inequality is strict is provided instead by $X = K(0, 1)$, $Y = K(0, 2)$ and $q = 2$.

Remark 5.6 Since stable ranks are measurable functions $[0, \infty) \to [0, \infty)$, there are many pseudometrics to compare them other than the interleaving distance $d_{\bowtie}$. In particular, one can consider the standard L^p-pseudometrics, here denoted by $d_p(f, g) := \left(\int_0^\infty |f(t) - g(t)|^p\, dt\right)^{\frac{1}{p}}$. As shown in Chachólski and Riihimaki (2020, Prop. 2.1), the stability theorem of hierarchical stabilization implies the following bounds for d_p:

$$c\, d(X, Y)^{\frac{1}{p}} \geq d_p\left(\widehat{\text{rank}}_d(X), \widehat{\text{rank}}_d(Y)\right),$$

for any persistence modules X and Y, where $c := \max\{\text{rank}(X), \text{rank}(Y)\}$ and d denotes any pseudometric between persistence modules. In this article we have chosen to work with the interleaving distance between Wasserstein stable ranks because of the strong stability result, expressed as a 1-Lipschitz condition. Lipschitz stability for Wasserstein distances other than W_1 can not be obtained for example by considering linear representations of persistence diagrams (Hofer et al. 2017; Adams et al. 2017; Chen et al. 2015; Kusano et al. 2017; Reininghaus et al. 2015) as proved in Theorem 6.3 in Skraba and Turner (2020). The trade-off between stability and the possibility of exploiting a Banach or Hilbert space structure is still to be explored.

5.3 Metric learning

We have defined distances $d^q_{\mathcal{S}p,C}$ between persistence modules, parametrized by q, p and by a contour C, and computable stable rank invariants with corresponding interleaving distances. These distances can be pulled back to compare persistence modules in Tame via the function $\widehat{\text{rank}}_d$, with $d = d^q_{\mathcal{S}p,C}$. Recalling that the stable ranks depend on the pseudometric $d^q_{\mathcal{S}p,C}$, we now turn to the question of how to choose p and C. The optimization of the parameter q is not relevant, since it determines a constant multiplicative factor to the distance of each pair of persistence modules. We thus fix $q = 1$ for a direct comparison with the original framework of noise systems.

For brevity, we write $d := d^1_{\mathcal{S}p,C}$ and $d_{\bowtie,p,C}(X, Y) := d_{\bowtie}(\widehat{\text{rank}}_d(X), \widehat{\text{rank}}_d(Y))$. The field of metric learning provides a variety of loss functions suited for different machine learning problems. For example, if we consider a simple binary classification problem we have a dataset of persistence modules $\{X_i\}_{i\in I}$ and the index set I is partitioned into two sets A and B, to represent the labeling. For this problem, a loss function (from Zhao and Wang 2019), designed to yield small intra-class distances and large inter-class distances can be formulated as:

$$\mathcal{L} = \frac{\sum_{i,j\in A}(d_{\bowtie,p,C}(X_i, X_j))^2}{\sum_{i\in A,j\in I}(d_{\bowtie,p,C}(X_i, X_j))^2} + \frac{\sum_{i,j\in B}(d_{\bowtie,p,C}(X_i, X_j))^2}{\sum_{i\in B,j\in I}(d_{\bowtie,p,C}(X_i, X_j))^2}. \tag{10}$$

In order to proceed we need to choose a family of contours that is practically searchable when minimizing the loss function above. We work with contours of distance type which are parametrized by densities (see Sect. 2.2). In turn, in order to use gradient optimization methods, we want the densities to be parametrized by a finite real-valued parameter vector. To this aim we choose as densities unnormalized Gaussian mixtures $f(x) = \sum_{i=1}^k \lambda_i \mathcal{N}(x \mid \mu_i, \sigma_i)$ for some chosen k, where $\mathcal{N}$ is Gaussian with mean μ_i and standard deviation σ_i, and $\lambda_1 = 1$.

In summary, the metric learning problem amounts to minimizing the loss function with respect to a parameter vector $\theta \in \mathbb{R}^{3k}$, i.e. $\theta = (\mu_1, \dots, \mu_k, \sigma_1, \dots, \sigma_k, \lambda_2, \dots, \lambda_k, p)$, designed to learn conjointly the parameter p and the parameters of the contour of the algebraic Wasserstein distance. The loss function is a simple function of the pairwise interleaving distances between Wasserstein stable ranks of persistence modules in the dataset. As can be seen in Proposition 5.3 and the expression (9), the interleaving distance between stable ranks is the L^∞ norm of differentiable functions with respect to θ and is therefore differentiable almost everywhere with respect to θ, implying the same behavior for the loss function. Hence the metric learning problem is amenable to gradient-based optimization methods such as gradient descent.

6 Examples of analyses with Wasserstein stable ranks

In a first experiment, we show how varying the parameter p affects the distance space of the Wasserstein stable ranks and can serve as a way to weight the importance of long bars versus short bars, for a set of synthetic persistence modules. In a second experiment, we illustrate on a real-world dataset how learning the parameter p together with the parameters of a contour can lead to more discriminative Wasserstein stable ranks in a classification problem.

6.1 Synthetic data

A straightforward way to apply persistent homology in the context of computer vision is to construct a complex (e.g. cubical complex) from the grid of pixels constituting an image. The complex is then filtered based on the grayscale intensity of the pixels (or based on the color channels for color images).

It is easy to see that, in this context, what should be considered as signal versus noise in a barcode representation of the data is highly dependent on the application. For example, for classification of handwritten digits from the MNIST dataset (Garin and Tauzin 2019; Turkeš et al. 2021) the dominant topological features are often the most discriminative (for instance the existence of a 1-dimensional cycle may be enough to distinguish between digits 0 and 1). On the other hand, in biomedical imaging (Chung et al. 2018; Qaiser et al. 2019) pathological states can translate into images with irregularities or lack of homogeneity, associated with high numbers of short-lived components as observed in Garside et al. (2019).

Inspired by these applications, we construct two much simpler synthetic datasets of images and associated persistence modules, with the goal of illustrating the effect of choosing the parameter p when using Wasserstein stable ranks. The parameter q is set to 1 and the contour is fixed to be the standard contour. In other words, we study the effect of the parameter p on how the function $\widehat{\text{rank}}_d$, with $d = d^1_{\mathcal{S}^p}$, maps persistence modules onto the space of stable ranks, endowed with the interleaving distance. Each dataset is composed of 100 images together with their class label, A or B. Each image is composed of one block of high-intensity pixels and a number of blocks of low-intensity pixels (while the size of the pixel blocks does not have a direct impact on the following persistent homology analysis, the high-intensity block is made larger for visual clarity, see Figs. 3, 4). The images are represented as cubical complexes on which super-level set filtration is performed and we analyze the H_0 barcodes obtained from this process. Since we use pixel intensity [0, 255] and super-level sets are used, the resulting filtration scale is $[255, -\infty)$. This is capped to the minimum pixel value, 0, and transformed as $255 - x$ to obtain a filtration scale [0, 255] as can be seen in the barcodes in Figs. 3, 4.

- In Dataset 1 the pixels in the high-intensity block have slightly higher intensity in images from class A (uniformly distributed between 245 and 255) compared to images of class B (between 200 and 210). The low-intensity blocks however follow the same distribution for images of both classes (the number of blocks is uniformly distributed between 50 and 100 and the intensity is between 1 and 10). Sample images and barcodes are shown in Fig. 3.
- In Dataset 2 on the other hand, the intensity of the high-intensity blocks follows the same distribution for both classes (uniformly distributed between 100 and 255). The number of low-intensity blocks however follows a different distribution for Class A (between 20 and 30) and Class B (between 120 and 130). Their intensity

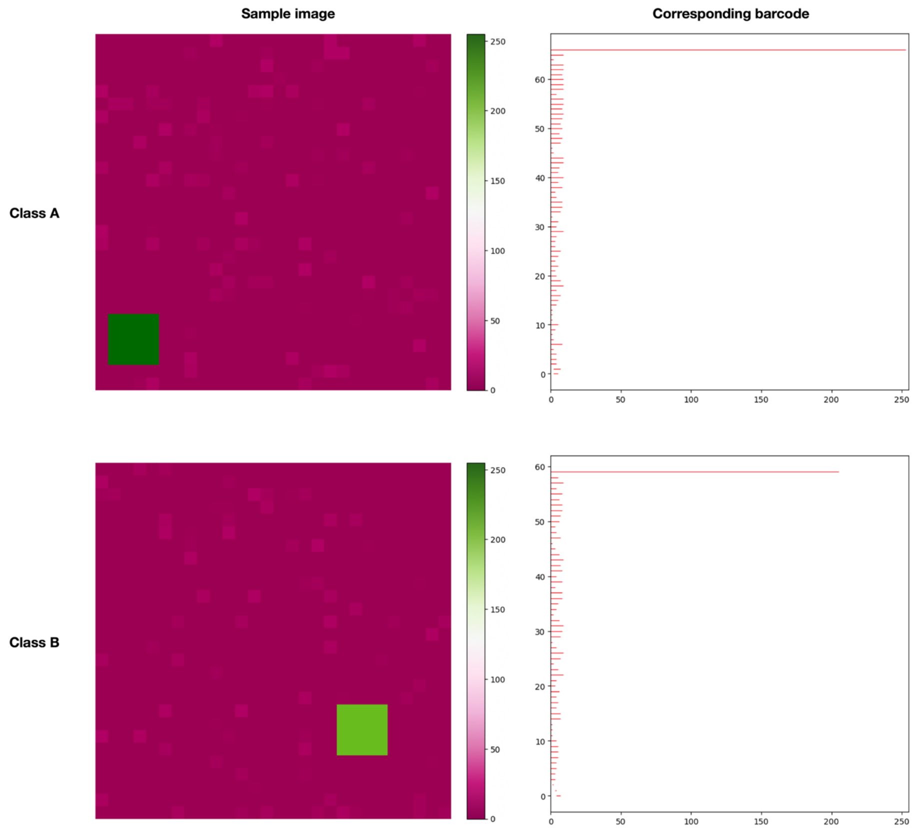


Fig. 3 Dataset 1. **Left:** Sample images from classes A and B. **Right**: H_0 barcodes corresponding to the sample images

is the same for both classes (between 1 and 10). Sample images and barcodes are shown in Fig. 4.

This construction induces distributions of barcodes where barcode features (i.e. length of longest bar or number of bars in a given length range) are expected to be statistically distinct or indistinguishable. In terms of the barcodes, for Dataset 1 the signal is by construction the single dominant topological feature (the long bar, whose length follows statistically different distributions between classes) while the noise is composed of the numerous short bars (corresponding to low intensity blocks, for which the number and intensity follows the same distribution in both classes).

In accordance with the intuition, for Dataset 1, choosing a value of $p = \infty$ when generating the stable ranks effectively "denoises" the barcodes and organizes the space of Wasserstein stable ranks in a way where stable ranks of the same class are close to each other in interleaving distance but far from elements of the other class. Stable ranks corresponding to $p = 1$ however fail to organize the corresponding distance space in this clear-cut way, being too sensitive to the noisy short bars in the barcodes. To illustrate this effect, in Fig. 5 we show the hierarchical clustering (with average

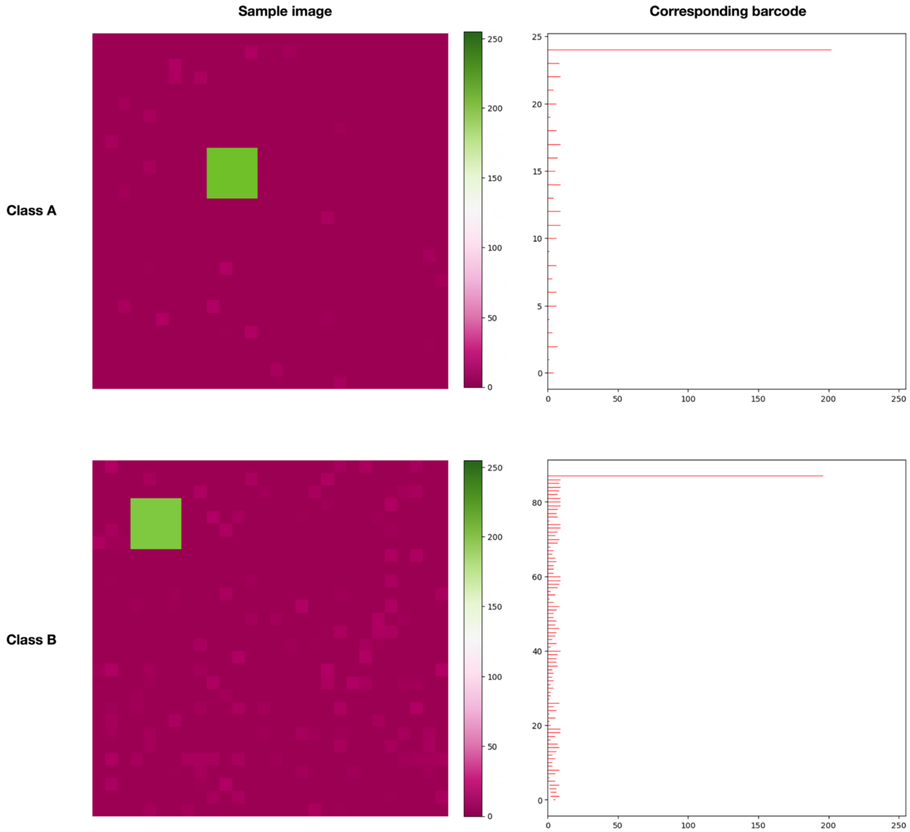


Fig. 4 Dataset 2. **Left:** Sample images from classes A and B. **Right**: H_0 barcodes corresponding to the sample images

linkage, similar results were observed for complete and single linkage) corresponding to the distance spaces of Wasserstein stable ranks for $p = 1$ and $p = \infty$.

On the contrary, for Dataset 2 the signal is by construction the number of short bars (numbers which follow statistically different distributions) while the noise is the single long bar (generated by blocks following the same distribution for both classes). In this case a choice of $p = 1$ organizes the space of stable ranks such that elements of the same class cluster together, while $p = \infty$, being too sensitive to the (for this dataset) noisy long bar, fails to do so. This is illustrated in Fig. 6.

While the effect of changing p on the structure of the distance space is clear for the parameters used to generate our artificial datasets, some class-based structure remains on a small scale, also when choosing $p = \infty$. By increasing the amount of noise it is however possible to induce e.g. a nearest neighbor classifier to perform arbitrary poorly for the $p = \infty$ while still distinguishing the classes for $p = 1$ (and vice versa for Dataset 1).

The choice of the parameter value p, which we have demonstrated can have a large impact, is essentially related to the underlying distance between persistence modules.

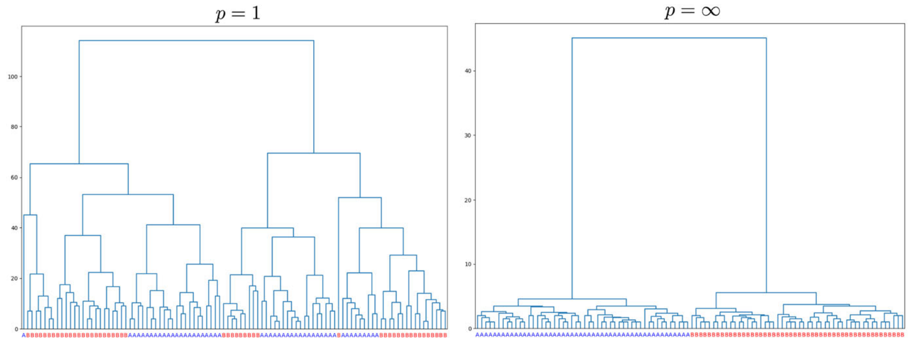


Fig. 5 Dataset 1. Hierarchical clustering on the Wasserstein stable ranks for $p = 1$ (left) and $p = \infty$ (right) with respect to the interleaving distance. The leaves (stable ranks in the dataset) are labeled and colored according to their class

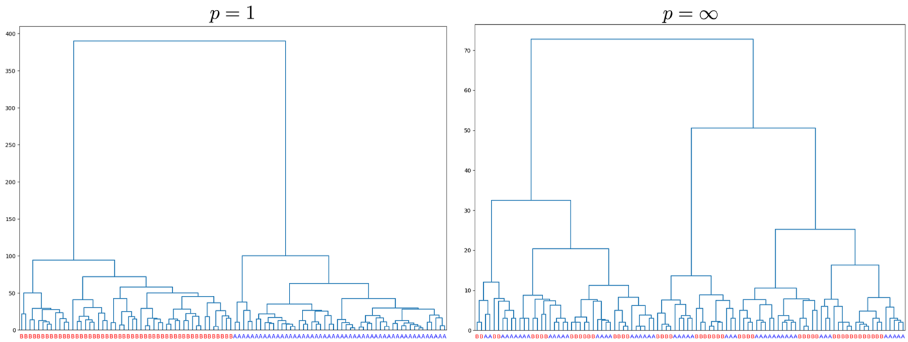


Fig. 6 Dataset 2. Hierarchical clustering on the Wasserstein stable ranks for $p = 1$ (left) and $p = \infty$ (right) with respect to the interleaving distance. The leaves (stable ranks in the dataset) are labeled and colored according to their class

Using Wasserstein-stable invariants however has computational advantages, facilitates learning the right parameters for a particular problem and allows for a richer use of machine learning methods as we illustrate in the next subsection on a real-world dataset.

6.2 Brain artery data

In Bullitt et al. (2010) a dataset of brain artery trees corresponding to 97 subjects aged 18 to 72 is introduced. Each data point is modeled as a tree embedded in $\mathbb{R}^3$. In Bendich et al. (2016) the dataset is further analyzed with persistent homology. To be able to apply a sublevel set filtration on the tree, a real-valued function is defined on the vertices as the height of the vertex in the 3D-embedding. This is extended to a function on the edges by taking the maximum value of the weights of the vertices connected by the edge. After applying persistent homology, each tree is represented by a vector containing the sorted lengths of the 100 longest bars in a barcode decomposition of the corresponding persistence module. This feature is further used to demonstrate, among

Table 1 Converged values for the parameters of the metric learning problem (mean and standard deviation over the LOOCV folds)

p	μ_1	μ_2	σ_1	σ_2	λ
1.10 ± 0.04	16.45 ± 0.75	81.88 ± 0.27	15.51 ± 0.45	4.55 ± 0.24	1.15 ± 0.05

other things, an age effect of brain artery structure, by showing that the projection of the vectors on the first principal component of the dataset is correlated with age.

The authors note that using vectors of sorted length was computationally more feasible than computing Wasserstein distances between the persistence diagrams and they are more amenable to statistical analysis. In addition, the authors observed that it was not necessary to use the whole vector of lengths to establish the correlation and in fact the topological features of medium length, rather than the longest ones, were the most discriminatory.

Analyzing the dataset with stable ranks offers computational and statistical advantages. Moreover, for this problem where the discriminative information is not contained in the most persistent feature, considering other distances than the bottleneck ($p = \infty$) and more generally tuning the parameter p might be beneficial. Finally, combining the tuning of the parameter p with a contour might increase the power of the method. Indeed the parameter p and the contour, intuitively are related to different features of a persistence barcode: while the parameter p globally weights the importance of long versus short bars as illustrated in Sect. 6.1, the contour allows to focus on the most informative filtration scales.

While we also study age effects of brain artery structure, we choose to binarize the problem by creating two classes of similar size: *young* ($age < 45$, 50 subjects) and *old* ($age \geq 45$, 47 subjects) and treat the problem as a classification.

We start by studying the effect of varying p alone. We compute the distances between Wasserstein stable ranks with standard contour. We classify the samples in the distance space thus obtained by using the k-nearest neighbors algorithm (Pedregosa et al. 2011) (the parameter k is chosen in a cross-validation procedure). Repeating this for various values of p we observe a difference in the resulting accuracy, plotted in Fig. 7, with the highest values obtained for p in the medium range (2–3). This is in line with the conclusion in Bendich et al. (2016) that the highest persistent features alone have a small distinguishing power, while medium sized bars reflect variations in brain artery trees within subject of different ages. In Bendich et al. (2016) only the length of bars in a barcode is used to compare barcodes of different classes. With our features parametrized by p and a contour C, we can take into consideration both the length of bars and their position in the parameter space.

We therefore next turn to the problem of learning the contour of the stable ranks as well. We use the metric learning method described in Sect. 5.3. Using leave-one-out cross-validation (LOOCV), for each training fold we learn the metric that optimally separates training samples from the two classes by minimizing the loss defined in (10). We then classify using KNN in the obtained distance space.

For metric learning, the contours are parametrized by densities which are unnormalized Gaussian mixtures with two components. The loss function is implemented

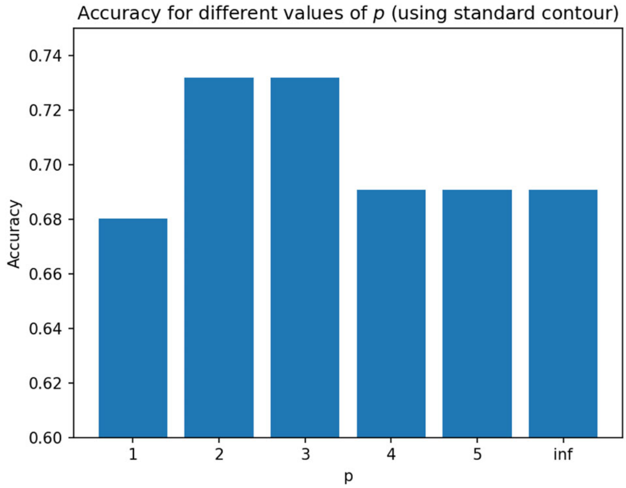


Fig. 7 Accuracy on the brain artery problem using distances between stable ranks and KNN, for standard contour and different values of p

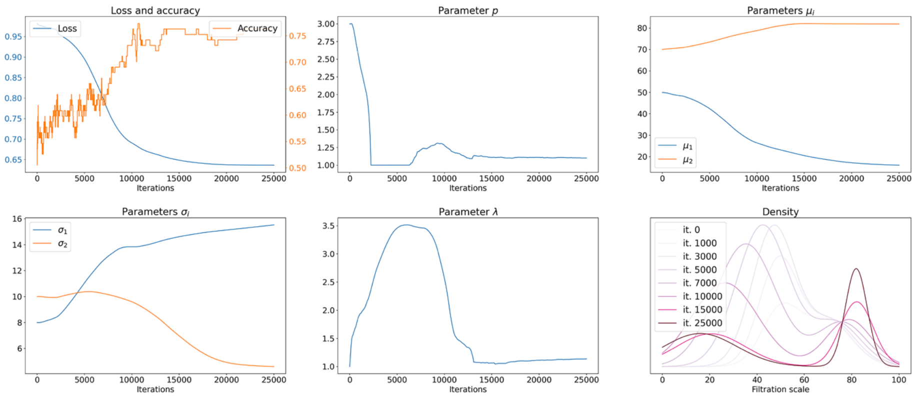


Fig. 8 Results for one example run of the metric learning optimization for Wasserstein stable ranks (see Sect. 5.3) over 25000 iterations. **Top Left:** Progression of the loss and the KNN training fold accuracy over the iterations. **Top Middle, Top Right, Bottom Left, Bottom Middle**: Progression of the parameters in $\theta = (\mu_1, \mu_2, \sigma_1, \sigma_2, \lambda_2, p)$ parametrizing Wasserstein stable ranks: p, mean μ_i, standard deviation σ_i and λ_2 respectively over the iterations. **Bottom Right**: Density at different iterations

in PyTorch (Paszke et al. 2019). After a random initialization of the parameters, projected gradient descent (to respect the constraints $p \geq 1, \lambda_i, \sigma_i > 0$) with momentum is used to achieve a lower loss. An example of an optimization on a training fold over 25000 iterations is shown in Fig. 8. The average and standard deviation of the optimized parameters, over the LOOCV folds can instead be found in Table 1.

The metric learning is effective in finding distances that improve the classification performance: running the optimization problem not only decreases the loss but also increases the corresponding classification accuracy (as is seen in Fig. 8 in the top left plot), reaching 76.3% with the parameter values summarized in Table 1.

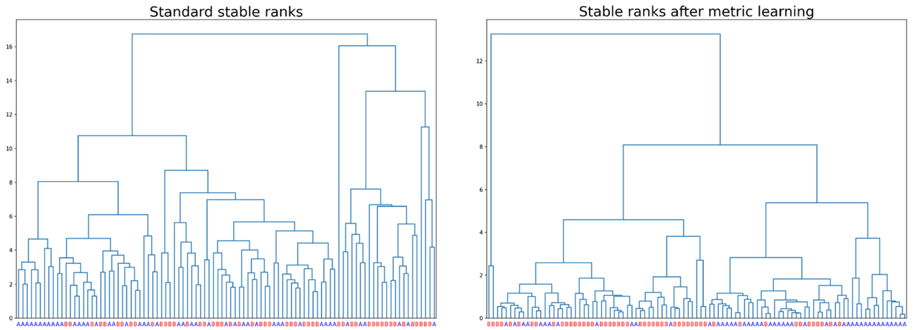


Fig. 9 Hierarchical clustering on the standard stable ranks (left) and the optimized stable ranks resulting from the metric learning problem (right) with respect to the interleaving distance. The leaves (stable ranks in the dataset) are labeled and colored according to their class (A is $age \geq 45$, B is $age < 45$)

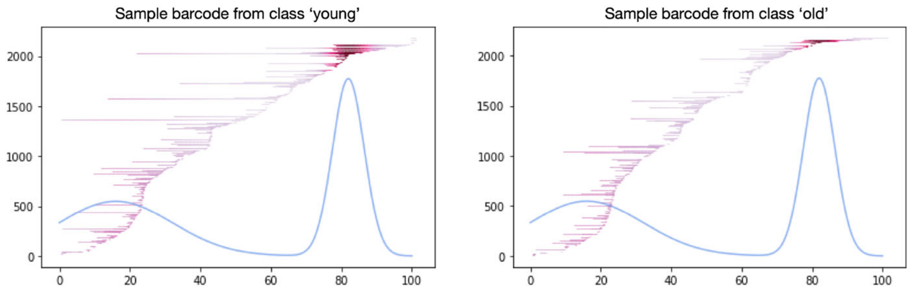


Fig. 10 Sample barcodes from the two classes with superposed learned density. Bars are colored according to the density

This is an improvement compared to the accuracies obtained at random initialization (between 44.3 and 71.1% for 10 random initializations of the parameters in Table 1), showing the benefit of learning, but also compared to the results obtained when only varying p and considering the standard contour in Fig. 7. It is thus when we learn both p and the contour that the best loss and corresponding classification accuracy is achieved.

In Fig. 9 we illustrate the effect of the metric learning by plotting the hierarchical clustering (with average linkage) corresponding to the standard stable ranks (i.e., with $p = \infty$ and standard contour) and to the optimized stable ranks. We see that the optimized stable ranks (with the exception of two outliers) group into two clusters: one with a majority of class A and the other with a majority of class B, while the pattern for standard stable ranks is less clear.

The optimal parameters found with the metric learning method are of interest because they allow to construct a distance space in which machine learning methods can be carried out, but they are also interpretable: they contain information about which features of the dataset are important to distinguish the two classes.

This is illustrated in Fig. 10 where two sample barcodes - one from each class - are displayed with the optimal density superposed and the bars colored according to the density. From the insight that some parts of the filtration scale are more important in distinguishing younger from older subjects, one may pursue the analysis by looking for characteristics of bars in that region of the barcode. One can also take the analysis a step further by looking at the object from which the filtered simplicial complex was created. In our case, since the filtration scale corresponds to the height (z-coordinate) in the 3D-embedding of the brain artery tree, one may for example investigate whether differences in brain artery between subjects of different ages in this particular region carries a biological meaning.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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Paper C

Barcode matchings and p-norms of persistence modules

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Barcode matchings and p-norms of persistence modules

Andrea Guidolin[i], Arthur Reidenbach[ii], Isaac Ren[iii], Martina Scolamiero[iii]

Abstract

We construct barcode matchings induced by morphisms of pointwise finite-dimensional persistence modules. To do this, we use an algebraic decomposition of the space of morphisms between persistence modules. Applying our induced matchings to arbitrary morphisms by the epi-mono factorization strategy of Bauer and Lesnick (2015), we recover the partial additive matching of Gonzalez-Diaz, Sorriano-Trigueros, and Torras-Casas (2025). We also study the interaction between our matching and a notion of cost for morphisms based on p-norms of persistence modules. We illustrate some consequences of our results on this cost in the context of algebraic stability of persistence barcodes, including a simple proof of the algebraic Wasserstein isometry of persistent homology.

C.1 Introduction

One of the central results in topological data analysis (TDA) is the barcode decomposition theorem, which states that, under some mild assumptions, every one-dimensional persistence module is isomorphic to a direct sum of basic

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interval modules, parameterized by intervals of the indexing poset. We can then represent the algebraic data of a persistence module by the combinatorial multiset of intervals. In this paper, we consider **pointwise finite-dimensional (p.f.d.) persistence modules** over a fixed field K, which are defined as the functors from the poset $(\mathbf{R}, \leq)$ to the category $\mathbf{Vect}_K$ of finite dimensional vector spaces. In particular, p.f.d. modules admit a barcode decomposition theorem [11, 5].

In this context, it is natural to ask whether morphisms of persistence modules also admit a combinatorial representation. The natural candidate for such representatives are (partial) **matchings**, or partial bijections, between barcodes. Barcode matchings are in fact quite central to the stability theory of persistent homology, as they form the basis of the definitions of the bottleneck and Wasserstein distances [12, Section VIII.2]. Barcode matchings induced by morphisms of persistence modules were studied previously by Bauer and Lesnick [3, 4], who recover various stability results for p.f.d. modules, by Skraba and Turner [22], who use induced matchings to show an isometry result between algebraic and combinatorial Wasserstein distances, and by Gonzalez-Diaz, Sorriano-Trigueros, and Torras-Casas [14, 16], who construct an additive barcode matching for persistence modules with finite barcodes.

Going back to the question of combinatorial representations of morphisms, Jacquard, Nanda, and Tillman provide a decomposition theorem for morphisms between persistence modules with finite barcodes that do not contain strictly nested bars [18, Theorem 5.3]. More precisely, given a morphism of persistence modules $f\colon \bigoplus_{i\in I} K(a_i, b_i) \to \bigoplus_{j\in J} K(c_j, d_j)$, where $K(a, b)$ denote the interval module with support $[a, b)$, if there is no pair of indices i, i' in I such that $[a_i, b_i)$ is a strict subset of $[a_{i'}, b_{i'})$, and no similar pair j, j' in J, then there exists a partial bijection $\mu \subseteq I \times J$ (i.e., each element of I and J appears at most once in μ) such that

$$f \cong \bigoplus_{(i,j)\in\mu} f_{i,j} \oplus \bigoplus_{i\in I\setminus\pi_I(\mu)} g_i \oplus \bigoplus_{j\in J\setminus\pi_J(\mu)} h_j, \tag{C.1}$$

where each f_{ij} is a nonzero morphism $K(a_i, b_i) \to K(c_j, d_j)$, each g_i denotes the zero morphism $K(a_i, b_i) \to 0$ and each h_j denotes the zero morphism $0 \to K(c_j, d_j)$. In [1], the authors (some of whom are authors of the current paper) call a morphism **bar-to-bar** if it admits a decomposition as in (C.1). Such bar-to-bar morphisms can be seen as an algebraic version of barcode matchings;

indeed, μ is the corresponding matching. In particular, the authors provide an algorithm for inducing bar-to-bar morphisms from mono- and epimorphisms.

C.1.1 Our contributions

In this paper, we continue the analysis of [1] by studying the association of matchings to morphisms. Following the strategy of [3], we induce a barcode matching for an arbitrary morphism $f\colon X \to Y$ by first finding an epi-mono factorization

$$f\colon X \xrightarrow{e} \mathrm{im}(f) \xrightarrow{m} Y$$

where $\mathrm{im}(f)$ denotes the image of f, $e\colon X \to \mathrm{im}(f)$ an epimorphism, and $m\colon \mathrm{im}(f) \to Y$ a monomorphism. We then compute the matchings of e and m associated to their induced bar-to-bar morphisms, and then compose them together to obtain a matching associated with f. Note that this depends *a priori* on the choice of epi-mono factorization; in practice, this is not the case for the induced matching of [3] but is the case for our matchings (Example C.43).

The key insight of this paper is that we can induce bar-to-bar morphisms by projecting morphisms down to a subspace of morphisms that only map nontrivially between bars with the same end-point: this is the **same-endpoint operator**, denoted by Δ (Definition C.25). Given a morphism f, Δf is a direct sum of morphisms between persistence modules with no nested bars. We prove, in the p.f.d. setting, that these morphisms are bar-to-bar, which implies that Δf is bar-to-bar (Theorem C.35). Notably, this simplifies the construction of induced bar-to-bar morphisms from [1]. As described above, this naturally induces a barcode matching $\mu_\Delta(f)$ (Definition C.41). In particular, this matching is additive: given morphisms f and g, $\mu_\Delta(f \oplus g) = \mu_\Delta(f) \oplus \mu_\Delta(g)$.

Our proof techniques rely on matrix representations of morphisms. To this end, we introduce the notion of labeled matrices and their multiplication (Definition C.12). In particular, to show that Δf is bar-to-bar, we reduce a matrix representation of Δf to a matrix with at most one nonzero coefficient in each row and column (Algorithm C.1). The induced matching is then read from the pivots of this reduced matrix representation. This matrix reduction is similar to [18, Theorem 5.3], the decomposition theorem for morphisms of persistence modules without nested bars.

C.1.2 Organization of the paper

In Section C.2, we define the notions used in this paper, including p.f.d. persistence modules, labeled matrices, barcode matchings, and dualization results.

In Section C.3, we introduce the notion of bar-to-bar morphisms and describe how to induce barcode matchings from an arbitrary morphisms using the same-end operator and epi-mono factorizations. In particular, we recover the $\mathscr{P}$-matching of [16] in Section C.3.4. Along the way, we obtain a simple algorithm for computing the barcode of the image of a morphism of persistence modules (Proposition C.46).

We apply bar-to-bar morphisms to (algebraic) Wasserstein distances in Section C.4. In Section C.4.1, we provide a simpler proof of [1, Theorem 3.14] (while also generalizing the result to p.f.d. modules), which states that, for a given monomorphism $f\colon X \hookrightarrow Y$, there exists a bar-to-bar monomorphism $f_b\colon X \hookrightarrow Y$ such that $\|\operatorname{coker} f_b\|_p \le \|\operatorname{coker} f\|_p$, where $\|-\|_p$ denotes the p-norm of p.f.d. persistence modules (Theorem C.61). As part of this new proof, we obtain an expression of the p-norm of a persistence module as an integral of the rank function (Proposition C.59). In Section C.4.2, we use the p-norm inequality result, and the dual result for epimorphisms, to prove the isometry between the algebraically defined and combinatorially defined Wasserstein distances (Theorem C.68).

C.2 Preliminaries

C.2.1 Persistence modules

We fix a field K and denote by $\mathbf{Vect}_K$ the category of finite dimensional vector spaces over K. Viewing $(\mathbf{R}, \le)$ as a poset category, a **pointwise finite-dimensional persistence module over** K is a functor $X\colon \mathbf{R} \to \mathbf{Vect}_K$, consisting therefore of a collection of finite dimensional vector spaces $\{X_t\}_{t\in\mathbb{R}}$ and linear maps between them $X_{s\le t}$ for $s \le t$, we call transition functions. We denote the category of pointwise finite-dimensional persistence modules and natural transformations between them by **Pfd**. In the following, pointwise finite-dimensional persistence modules are referred to as persistence modules for brevity. The category **Pfd** has biproducts (i.e. direct sums), which are computed pointwise[1].

In this setting, interval modules are not parameterized by the real line, but rather by an extension of the real line:

C.1 **Definition** (Decorated real line and intervals). This definition is adapted from [3, §2.3]. We consider the **decorated real line**

$$\mathbb{E} := \mathbf{R} \times \{-,+\} \cup \{(-\infty,+),(+\infty,-)\} \subseteq [-\infty,+\infty] \times \{-,+\}$$

equipped with the following lexicographic order: for (s,σ) and (t,τ) in $\mathbb{E}$, we have $(s,\sigma) < (t,\tau)$ if $s < t$ in $[-\infty,+\infty]$ or if $s = t$, $\sigma = -$, and $\tau = +$. We denote $(t,-)$ and $(t,+)$ in $\mathbb{E}$ by t^- and t^+, respectively, except when $t = \pm\infty$, where we omit the sign.

Ordered pairs of elements of $\mathbb{E}$ are in one-to-one correspondence with all intervals on the real line: for $s^\sigma < t^\tau$ in $\mathbb{E}$, we define the interval

$$\langle s^\sigma, t^\tau \rangle := \{u \in \mathbf{R} \mid s < u < t \text{ or } s^\sigma = u^- \text{ or } t^\tau = u^+\}. \qquad \blacksquare$$

In particular, for $s < t$ in $\mathbf{R}$, we have the following equality of intervals:

$$\begin{aligned} \langle s^-, t^- \rangle &= [s,t), & \langle s^-, t^+ \rangle &= [s,t], \\ \langle s^+, t^- \rangle &= (s,t), & \langle s^+, t^+ \rangle &= (s,t]. \end{aligned}$$

C.2 **Definition** (Overlapping intervals). Let $a,b,c,d \in \mathbb{E}$. Following [4], we say that **the interval** $\langle a,b \rangle$ **overlaps the interval** $\langle c,d \rangle$ **above** (and antisymmetrically, $\langle c,d \rangle$ **overlaps** $\langle a,b \rangle$ **below**) if $c \le a < d \le b$. $\blacksquare$

C.3 **Definition** (Interval modules). Let $a < b$ be elements of $\mathbb{E}$. A **bar** (or **interval module**) with **start-point** a and **end-point** b, denoted as $K(a,b)$, is the persistence module defined as follows: for any u in $\mathbf{R}$,

$$K(a,b)_u := \begin{cases} K & \text{if } u \in \langle a,b \rangle, \\ 0 & \text{otherwise,} \end{cases}$$

[1]It is a general fact that limits and colimits in functor categories are computed pointwise; **Pfd** is a functor category and biproducts are both limits and colimits.

and for any $u \leq v$ in $\mathbf{R}$,

$$K(a,b)_{u\leq v} := \begin{cases} \mathrm{id}_K & \text{if } u,v \in \langle a,b\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the **support** of $K(a,b)$, i.e. the real values where the functor is nonzero, is the interval $\langle a,b\rangle$. ∎

C.4 **Remark.** We observe that there exists a nonzero morphism $K(a,b) \to K(c,d)$ if, and only if, $\langle a,b\rangle$ overlaps $\langle c,d\rangle$ above. ∎

In order to better manage these interval modules, we will be considering the following subcategory of **Pfd**. Recall that a **multiset** is a set S equipped with a **multiplicity** $m_S\colon S \to \mathbf{N}$, and that when there is no ambiguity, S and m_S may be used interchangeably.

C.5 **Definition** (Subcategory of bars). We denote by **Bar** the full subcategory of **Pfd** whose objects are the (potentially infinite) direct sums of bars $\bigoplus_{i\in I} K(a_i,b_i)$. We call this the **category of bars**.

Given an object $X = \bigoplus_{i\in I} K(a_i,b_i)$ of **Bar**, we define its **barcode** as a representation of the multiset $\mathscr{B}(X) := \{\langle a_i,b_i\rangle\}_{i\in I}$ of intervals appearing in X (recall the definition of multiset representations from [4]). ∎

C.6 **Definition.** Let X be a persistence module in **Pfd**. A **barcode decomposition** of X is the given of an object $\bigoplus_i K(a_i,b_i)$ in **Bar** and an isomorphism $\varphi_X\colon X \cong \bigoplus_i K(a_i,b_i)$. ∎

C.7 **Remark.** The decomposition theorem [11, 5] states that every persistence module X in **Pfd** admits a barcode decomposition $\varphi_X\colon X \cong \bigoplus_i K(a_i,b_i)$. While the isomorphism φ_X is not unique, the direct sum of bars $\bigoplus_i K(a_i,b_i)$ is, up to reordering of the bars. In particular, we can speak of the **barcode** of an object in **Pfd** as well. ∎

C.8 **Remark.** Every persistence module in **Pfd** has at most countably many bars $K(s,t)$ with $s < t \in E$ in its barcode decomposition. In other words, up to an **ephemeral** summand (see [8]), which is a direct sum of bars of the form $K(s^-,s^+)$ supported on $\{s\}$, every persistence module X in **Pfd** has at most countably many bars. Moreover, for each $s \in \mathbf{R}$, X has a finite number of bars

$K(s^-, s^+)$, since the dimension of X_s is finite. As a consequence, for all $b \in \mathbb{E}$, X only has countably many bars with start-point b, and likewise countably many with end-point b. ■

C.9 **Proposition.** *The categories* **Bar** *and* **Pfd** *are equivalent, given the axiom of choice.*

Proof. We observe that the inclusion functor **Bar** $\hookrightarrow$ **Pfd** is fully faithful as the inclusion of a full subcategory and is essentially surjective by the decomposition theorem. Using the axiom of choice, we conclude that the inclusion functor is an equivalence of categories: see [23, Lemma 02C3]. □

Recall from [1] that a persistence module X in Pfd is **tame** if there exist real numbers $0 = t_0 < t_1 < \cdots < t_k$ such that the transition function $X_{s\leq t}$ is a non-isomorphism only if $s < t_i \leq t$ for some $i \in \{1, \ldots, k\}$. All the bars in the barcode decompositon of a tame persistence module are of the form $\langle s^-, t^- \rangle = [s, t)$.

As a notion of *size* for persistence modules, we use the p-norm first introduced in [9] and [22], also studied in [1] for tame persistence modules.

C.10 **Definition.** For $p \in [1, \infty]$, the ***p*-norm** of a persistence module with barcode decomposition $X \cong \bigoplus_i K(a_i, b_i)$ is:

$$||X||_p = \begin{cases} \left(\sum_i (b_i - a_i)^p\right)^{\frac{1}{p}} & \text{for } p \in [1, \infty), \\ \sup_i\{(b_i - a_i)\} & \text{for } p = \infty. \end{cases}$$ ■

By Remark C.8, the p-norm of a persistence module in **Pfd** involves an infinite sum with at most countably many positive summands. Following [22] (Definition 2.11) we say that a persistence module X has *bounded p-norm* if $||X||_p^p < \infty$.

C.2.2 Morphisms and matrices

One advantage of **Bar** over **Pfd** is that we can conveniently, and not just up to isomorphism, represent morphisms as (possibly infinite) matrices. Let $X = \bigoplus_{i\in I} K(a_i, b_i)$ and $Y = \bigoplus_{j\in J} K(c_j, d_j)$ be objects of **Bar**. Note that $\hom(X, Y)$ is a vector space, possibly infinite-dimensional. Moreover, given the canonical isomorphisms

$$\hom\Big(\bigoplus_{i\in I} K(a_i,b_i), \bigoplus_{j\in J} K(c_j,d_j)\Big) \cong \prod_{i\in I}\bigoplus_{j\in J} \hom(K(a_i,b_i),K(c_j,d_j)) \tag{C.2}$$

and

$$\hom(K(a_i,b_i),K(c_j,d_j)) \cong \begin{cases} K & \text{if } \langle a_i,b_i\rangle \text{ overlaps } \langle c_j,d_j\rangle \text{ above,} \\ 0 & \text{otherwise,} \end{cases} \tag{C.3}$$

we can then represent each morphism $f \in \hom(X,Y)$ with a matrix $F \in K^{I\times J}$, where $F_{ij} = 0$ if $\langle a_i,b_i\rangle$ does not overlap $\langle c_j,d_j\rangle$ above.

C.11 **Definition** (Column-finite matrices). A matrix $M \in K^{I\times J}$ with rows indexed by I and columns indexed by J is **column-finite** if each column only has a finite number of nonzero coefficients. ■

C.12 **Definition** (Labeled matrices). A **labeled matrix** is a (potentially infinite) matrix $A \in K^{I\times J}$ where I and J are the totally ordered indexing sets for rows and columns, respectively, and where each row and column is labeled by an interval of $\mathbf{R}$ such that, for all $(i,j) \in I\times J$, writing $\langle a_i,b_i\rangle$ for the row label and $\langle c_j,d_j\rangle$ for the column label, A_{ij} is nonzero only if $\langle c_j,d_j\rangle$ overlaps $\langle a_i,b_i\rangle$ above.

Let $A \in K^{I\times J}$ and $B \in K^{J\times L}$ be labeled, column-finite matrices where the columns of A and the rows of B have the same labels. The **overlap multiplication** of A and B, written as the usual product AB, is defined by, for all $(i,\ell) \in I\times L$,

$$(AB)_{i\ell} = \sum_{j\in J} A_{ij}B_{j\ell}\mathbb{I}\{a_i \le e_\ell < b_i \le f_\ell\}, \tag{C.4}$$

where the label of row i in A is $\langle a_i,b_i\rangle$ and the label of column ℓ in B is $\langle e_\ell,f_\ell\rangle$, and $\mathbb{I}\{a_i \le e_\ell < b_i \le f_\ell\} = 1$ if the inequalities hold and 0 otherwise. ■

C.13 **Remark.**

1. We check that overlap multiplication is associative, with the usual identity matrix as the unit.
2. The reason we assume that F and G are column-finite is to ensure that the sums in (C.4) are well-defined. There are other settings where the sums are well-defined, e.g. for row-finite matrices, but we will only need the result for column-finite matrices.

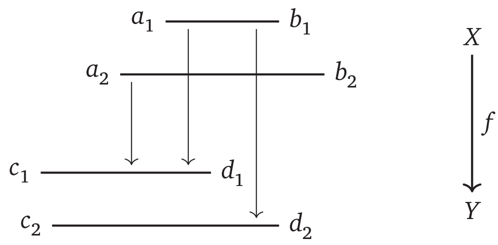


Figure C.1: Representation of a morphism $f\colon K(a_1,b_1) \oplus K(a_2,b_2) \to K(c_1,d_1) \oplus K(c_2,d_2)$ in **Bar**. An arrow between two bars $K(a_i,b_i)$ and $K(c_j,d_j)$ corresponds to the projection of f to $\hom(K(a_i,b_i),K(c_j,d_j))$; the morphism is represented by its nonzero projections.

3. When every interval indexed by L overlaps every interval indexed by I, overlap multiplication corresponds exactly to standard matrix multiplication. For example, this is the case when every interval has the same end-point. ■

C.14 **Definition** (Matrix representations of bar morphisms). Let $f\colon \bigoplus_{j\in J} K(a_j,b_j) \to \bigoplus_{i\in I} K(c_i,d_i)$ be a morphism in **Bar** (note the swapped indices). An **ordered matrix representation of** f is a labeled matrix $(F_{ij})_{i\in I, j\in J}$ where

1. for all $i \in I$ and $j \in J$, the entry F_{ij} is the projection of the morphism f to $\hom(K(a_j,b_j),K(c_i,d_i))$ via the canonical isomorphisms (C.2) and (C.3),
2. I and J are ordered in a way compatible with the product order on the pairs (a_j,b_j) and (c_i,d_i), and
3. row i is labeled by $\langle c_i,d_i\rangle$ and column j is labeled by $\langle a_j,b_j\rangle$. ■

C.15 **Example.** For example, the only ordered matrix representation of the morphism represented in Figure C.1 is:

$$F = \begin{array}{c} \\ \langle c_1,d_1\rangle \\ \langle c_2,d_2\rangle \end{array} \begin{array}{c} \begin{array}{cc} \langle a_1,b_1\rangle & \langle a_2,b_2\rangle \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \qquad ■$$

Under certain conditions, the composition of morphisms can also be represented by matrices.

C.16 **Proposition.** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in* **Bar**, *and F and G matrix representations of theirs. Suppose that F and G are column-finite. Then* *the overlap multiplication GF (Definition C.12) is a matrix representation of $g \circ f$.*

Proof. Write $X = \bigoplus_{\ell \in L} K(u_\ell, v_\ell)$, $Y = \bigoplus_{j \in J} K(c_j, d_j)$, and $Z = \bigoplus_{i \in I} K(a_i, b_i)$. Denote by $f_{j\ell}$ the restriction of f to $\hom(K(u_\ell, v_\ell), K(c_j, d_j))$ via the isomorphism (C.2), and similarly for g_{ij} and $(g \circ f)_{i\ell}$. Note the swapped indices, which are chosen to match matrix notations. Then, for all $(i, \ell) \in I \times L$, we have the equality of morphisms

$$(g \circ f)_{i\ell} = \sum_{j \in J} g_{ij} \circ f_{j\ell}.$$

Denote by $H \in K^{I \times L}$ the matrix representation of $g \circ f$ with columns in the same order as F and rows in the same order as G. From the previous equality, the isomorphism (C.3), and the definition of overlap multiplication, we deduce that

$$H_{i\ell} = \sum_{j \in J} G_{ij} F_{j\ell} \mathbb{I}\{a_i \leq u_\ell < b_i \leq v_\ell\} = (GH)_{i\ell},$$

and this concludes the proof. □

C.2.3 Matchings

Set-theoretic **matchings** are defined as partial injective functions between sets [4], but we will use the following equivalent definition in order to more easily generalize to multiset matchings.

C.17 **Definition** (Matchings). Let I and J be sets. A (**set-theoretic partial**) **matching between** I **and** J is a subset $\mu \subseteq I \times J$ such that the projections π_I and π_J onto I and J, respectively, are injective. We denote by **Match**(I, J) the set of matchings between I and J. ■

We use the definitions of the category **Mch** (including composition of matchings $\circ$), **multiset representations**, **barcodes**, and the category **Barc** (including the overlap composition of matchings $\bullet$) from [4]. We also define multiset matchings:

C.18 **Definition** (Multiset matchings [16, Definition 2.4]). Let (S, m_S) and (T, m_T) be two multisets. A (**partial multiset**) **matching between** S **and** T is a multiset (M, m_M) where $M = S \times T$ and, for all $s \in S$ and $t \in T$,

$$\sum_{t' \in T} m_M(s, t') \leq m_S(s) \quad \text{and} \quad \sum_{s' \in S} m_M(s', t) \leq m_T(t).$$

We denote by $\mathbf{Match}(S,T)$ the set of matchings between S and T. ■

In particular, this definition coincides with the set-theoretic one when S and T are sets (m_S and m_T bounded above by 1).

C.19 **Definition** (Matching costs). Let S and T be multisets of intervals of the real line, $M \in \mathbf{Match}(S,T)$ a matching, and $p \in [1,\infty]$. The ***p*-cost of** M is

$$c_p(M) := \left(\sum_{(\langle s^\sigma,t^\tau\rangle,\langle s'^{\sigma'},t'^{\tau'}\rangle)\in M} m_M(\langle s^\sigma,t^\tau\rangle,\langle s'^{\sigma'},t'^{\tau'}\rangle)(|s-s'|^p+|t-t'|^p) \right)^{\frac{1}{p}}.$$

In other words, $c_p(M)$ is the p-norm of the vector of lengths of intervals in the symmetric differences of matched intervals in M. ■

C.2.4 Dualization of persistence modules and matchings

In this paper, we will mainly prove results for monomorphisms in **Pfd**. Analogous results hold for epimorphisms by a duality argument. Specifically, we consider the contravariant dualization endofunctor $(\cdot)^*\colon$ **Pfd** $\to$ **Pfd** defined by [3, Section 3.4], which send a map $f\colon X\to Y$ to the **dual map** $f^*\colon Y^*\to X^*$. In particular, this functor it sends epimorphisms to monomorphisms, and vice versa. It also sends the bar $K(a,b)$ to the bar $K(-b,-a)$, where the negative sign acts on the decorated real line $\mathbb{E}$ by $-(t^\tau) = (-t)^\tau$.

We also define the dual of a matching as follows.

C.20 **Definition.** Let X and Y be persistence modules in **Pfd** and μ a matching in $\mathbf{Match}(\mathscr{B}(X),\mathscr{B}(Y))$. The **dual of** μ is defined as the matching μ^*

$$\mu^* := \{(\langle -d,-c\rangle,\langle -b,-a\rangle) \mid (\langle a,b\rangle,\langle c,d\rangle)\in\mu\}$$

in $\mathbf{Match}(\mathscr{B}(Y^*),\mathscr{B}(X^*))$. ■

C.3 Bar-to-bar morphisms and induced matchings

C.3.1 Bar-to-bar morphisms

C.21 **Definition.** A morphism $f\colon X\to Y$ of persistence modules in **Bar** is **strictly bar-to-bar** if, writing $X=\bigoplus_{i\in I}K(a_i,b_i)$ and $Y=\bigoplus_{j\in J}K(c_j,d_j)$, there exists a (set-theoretic) matching $\mu\in\mathbf{Match}(I,J)$ such that

$$f = \bigoplus_{(i,j)\in\mu} f_{i,j} \oplus \bigoplus_{i\in I\setminus\pi_I(\mu)} g_i \oplus \bigoplus_{j\in J\setminus\pi_J(\mu)} h_j, \tag{C.5}$$

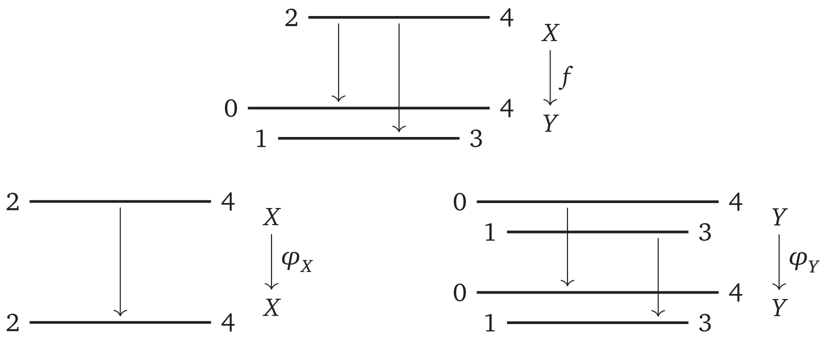


Figure C.2: **Top:** example of a non bar-to-bar morphism $f : X \to Y$. **Bottom:** the only valid automorphisms $\varphi_X : X \to X$ and $\varphi_Y : Y \to Y$. In both cases, the image of the restriction of the automorphism to a bar is contained within that same bar. Consequently, they do not alter the morphism when composed.

where each f_{ij} is a nonzero morphism $K(a_i, b_i) \to K(c_j, d_j)$, each g_i denotes the zero morphism $K(a_i, b_i) \to 0$ and each h_j denotes the zero morphism $0 \to K(c_j, d_j)$. The matrix associated to a strictly bar-to-bar morphism has at most one non zero entry for each column and one non zero entry for each row.

A morphism $f\colon X \to Y$ in **Pfd** is **bar-to-bar** if there exist barcode decompositions of X and Y such that the associated morphism in **Bar** is strictly bar-to-bar. In other words, there exist barcode decompositions $(\varphi_X, \bigoplus_i K(a_i, b_i))$, $(\varphi_Y, \bigoplus_j K(c_j, d_j))$ of X and Y respectively and $g\colon \bigoplus_i K(a_i, b_i) \to \bigoplus_j K(c_j, d_j)$ strictly bar-to-bar such that the following diagram commutes:

$$\begin{tikzcd}
\bigoplus_i K(a_i, b_i) \arrow[d, "g"'] & X \arrow[l, "\varphi_X"'] \arrow[d, "f"] \\
\bigoplus_j K(c_j, d_j) & Y \arrow[l, "\varphi_Y"]
\end{tikzcd}$$

■

Not every morphism is bar-to-bar: see Figure C.2 for an example where no isomorphisms would make $f\colon X \to Y$ strictly bar-to-bar.

C.22 **Proposition.** *The composition of two strictly bar-to-bar monomorphisms is strictly bar-to-bar.*

Proof. Consider strictly bar-to-bar morphisms $f\colon X \to Y$ and $f'\colon Y \to Z$ with decompositions

$$f = \bigoplus_{(i,j)\in\mu} f_{ij} \oplus \bigoplus_{i\in I\setminus\pi_I(\mu)} g_i \oplus \bigoplus_{j\in J\setminus\pi_J(\mu)} h_j$$
$$\text{and} \quad f' = \bigoplus_{(j,\ell)\in\mu'} f'_{j\ell} \oplus \bigoplus_{j\in J\setminus\pi_J(\mu')} g'_j \oplus \bigoplus_{\ell\in L\setminus\pi_L(\mu)} h'_\ell,$$

where in particular $\mu \in \mathbf{Match}(I,J)$ and $\mu' \in \mathbf{Match}(J,L)$ are set-theoretic matchings. Their composition $f' \circ f$ has the decomposition

$$f' \circ f = \bigoplus_{\substack{(i,\ell)\in\mu'\circ\mu \\ (i,j)\in\mu,\,(j,\ell)\in\mu'}} f'_{j\ell} \circ f_{ij} \oplus \bigoplus_{i\in I\setminus\pi_I(\mu)} g_i \oplus \bigoplus_{\ell\in L\setminus\pi_L(\mu')} h'_\ell,$$

where $\mu' \circ \mu$ is the set-theoretic matching that results from composing μ and μ' as multiset matchings, and the j in the first sum is unique, given $(i,\ell) \in \mu' \circ \mu$. We observe that $f'_{j\ell} \circ f_{ij}$ is nonzero if, and only if, (i,ℓ) is in the overlap composition $\mu' \bullet \mu$ (see Remark C.4). If not, then $i \in I \setminus \pi_I(\mu' \bullet \mu)$, $\ell \in L \setminus \pi_L(\mu' \bullet \mu)$, and $f'_{j\ell} \circ f_{ij}$ is equal to the sum of zero maps $(K(a_i, b_i) \to 0) \oplus (0 \to K(e_\ell, f_\ell))$. Thus we can write

$$f' \circ f = \bigoplus_{\substack{(i,\ell)\in\mu'\bullet\mu \\ (i,j)\in\mu,\,(j,\ell)\in\mu'}} f'_{j\ell} \circ f_{ij} \oplus \bigoplus_{i\in I\setminus\pi_I(\mu'\bullet\mu)} g_i \oplus \bigoplus_{\ell\in L\setminus\pi_L(\mu'\bullet\mu)} h'_\ell,$$

which is an equation of the form (C.5), and this concludes the proof. □

C.23 **Remark.** The composition of arbitrary bar-to-bar monomorphisms is not necessarily bar-to-bar. For example, consider the following composition of morphisms (see Figure C.3):

$$K(1,4) \oplus K(2,3) \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K(1,4) \oplus K(1,3) \xrightarrow{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} K(0,4) \oplus K(1,3),$$

where the sign decorations of the start- and end-points are not important. Clearly the left map is bar-to-bar. The right map is bar-to-bar by doing a base change on the middle persistence module. However, the composition is a morphism between persistence modules with trivial automorphism groups, but its matrix representation looks like that of the right morphism, and therefore the composition morphism is not bar-to-bar. ■

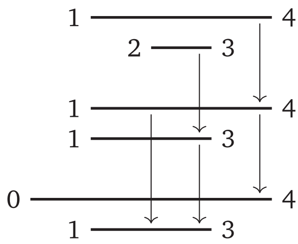


Figure C.3: Representation of the morphisms in **Bar** of Remark C.23.

We can view the matching μ of Definition C.21 as a matching of barcodes. Specifically, μ induces the multiset matching (M, m_M) where $M = \mathscr{B}(X) \times \mathscr{B}(Y)$ and, for all $(\langle a,b\rangle, \langle c,d\rangle) \in M$,

$$m_M(\langle a,b\rangle, \langle c,d\rangle) := \#\{(i,j) \in \mu \mid (a_i, b_i) = (a,b) \text{ and } (c_j, d_j) = (c,d)\}.$$

It turns out that this matching is uniquely determined as soon as we have a bar-to-bar morphism in **Pfd**:

C.24 **Proposition.** *Let $f\colon X \to Y$ be a bar-to-bar morphism in* **Pfd***, and let:*

- *$\varphi_X\colon X \cong \bigoplus_{i\in I} K(a_i, b_i)$ and $\varphi_Y\colon Y \cong \bigoplus_{j\in J} K(c_j, d_j)$ be barcode decompositions;*
- *$g\colon \bigoplus_{i\in I} K(a_i, b_i) \to \bigoplus_{j\in J} K(c_j, d_j)$ be a strictly bar-to-bar morphism in* **Bar** *isomorphic to f (which exists by definition);*
- *μ be a matching in $\mathbf{Match}(I,J)$ associated to g (see Definition C.21).*

Then the multiset matching $M \in \mathbf{Match}_0(\mathscr{B}(X), \mathscr{B}(Y))$ induced by μ does not depend on φ_X, φ_Y, g, nor μ.

Proof. We view f as an object in the category $\mathrm{Fun}(\mathbf{R} \times \{0 < 1\}, \mathbf{Vect}_K)$. As such a representation, the decomposition (C.5) of g becomes

$$f \cong g \cong \bigoplus_{(i,j)\in\mu} \mathbf{R}^{\langle a_i,b_i\rangle}_{\langle c_j,d_j\rangle} \oplus \bigoplus_{i\in I\smallsetminus \pi_I(\mu)} \mathbf{I}^{\langle a_i,b_i\rangle}_0 \oplus \bigoplus_{j\in J\smallsetminus \pi_J(\mu)} \mathbf{I}^0_{\langle c_j,d_j\rangle},$$

where the indecomposables $\mathbf{R}^*_*$ and $\mathbf{I}^*_*$ are adapted from [18, Definition 5.2] and defined as follows:

- $\mathbf{R}^{\langle a_i,b_i\rangle}_{\langle c_j,d_j\rangle}$, where $\langle a_i, b_i\rangle$ overlaps $\langle c_j, d_j\rangle$ above, is the representation of a nonzero morphism $K(a_i, b_i) \to K(c_j, d_j)$.

- $\mathbf{I}_0^{\langle a_i, b_i \rangle}$ is the representation of the zero morphism $K(a_i, b_i) \to 0$.
- Similarly, $\mathbf{I}^0_{\langle c_j, d_j \rangle}$ is the representation of the zero morphism $0 \to K(c_j, d_j)$.

By the Krull-Remak-Schimdt-Azumaya theorem [2] (see also [5]), this decomposition is unique up to permutation of the indecomposables. This implies that the matching μ is unique up to permutations of indices i, i' in I such that $a_i = a_{i'}$ and $b_i = b_{i'}$, and similarly for J. In particular, the induced matching M on barcodes is unique. The fact that M is an overlap matching is clear from its construction, and this concludes the proof. □

C.3.2 The same-endpoint operator

We define the same-endpoint operator on direct sums of bars, and then show that it is well-defined up to isomorphism on arbitrary persistence modules in **Pfd**.

C.25 **Definition** (same-endpoint operator Δ). We define Δ as the operator $\mathbf{Bar} \to \mathbf{Bar}$ that is the identity on objects and the projection

$$\begin{aligned}\hom(X, Y) &\cong \prod_{i \in I} \bigoplus_{j \in J} \hom(K(a_i, b_i), K(c_j, d_j)) \\ &\longrightarrow \prod_{i \in I} \bigoplus_{\substack{j \in J \\ b_i = d_j}} \hom(K(a_i, b_i), K(c_j, d_j)) \\ &\hookrightarrow \hom(X, Y)\end{aligned}$$

on morphisms, where $X = \bigoplus_i K(a_i, b_i)$ and $Y = \bigoplus_j K(c_j, d_j)$. We call Δ the **same-endpoint operator**. ■

C.26 **Remark.** The first isomorphism and the last inclusion are canonical. Thus the operator Δ is well-defined and does not depend on any arbitrary choices. ■

C.27 **Example.** Figure C.4 shows an example of applying the same-endpoint operator to a monomorphism in **Bar**. ■

From the definition of Δ as a linear projection on hom-spaces, we immediately get the following results.

C.28 **Proposition.** *1. The operator Δ is linear on hom-spaces. That is, for morphisms $f, g \colon X \to Y$, $\Delta(f + g) = \Delta f + \Delta g$.*

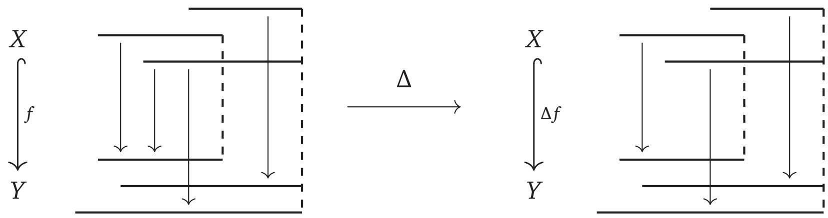


Figure C.4: same-endpoint operator applied to a monomorphism $f\colon X \hookrightarrow Y$ in **Bar**. The resulting monomorphism Δf is obtained by setting to zero all the nonzero projections between bars with different end points.

2. *The operator Δ is additive. That is, for morphisms $f\colon X \to Y$ and $g\colon X' \to Y'$, we have the equality $\Delta(f \oplus g) = \Delta f \oplus \Delta g$ as morphisms $X \oplus X' \to Y \oplus Y'$.*

C.29 **Proposition.** *The operator Δ is a functor.*

Proof. Clearly $\Delta \mathrm{id} = \mathrm{id}$. For compatibility with composition, consider the composable morphisms $f\colon \bigoplus_i K(a_i, b_i) \to \bigoplus_j K(c_j, d_j)$ and $g\colon \bigoplus_j K(c_j, d_j) \to \bigoplus_k K(s_k, t_k)$. We write f_{ij} for the corresponding morphism

$$\bigoplus_i K(a_i, b_i) \xrightarrow{p_i} K(a_i, b_i) \to K(c_j, d_j) \xrightarrow{\iota_j} \bigoplus_j K(c_j, d_j)$$

and similarly g_{jk} and $(g \circ f)_{ik}$. Then

$$g \circ f = \sum_{i,j,k} g_{jk} \circ f_{ij} = \sum_{\substack{i,j,k \\ b_i \geq d_j \geq t_k}} g_{jk} \circ f_{ij} \quad \text{and} \quad (g \circ f)_{ik} = \sum_{\substack{j \\ b_i \geq d_j \geq t_k}} g_{jk} \circ f_{ij}$$

because all other summands are 0. Moreover,

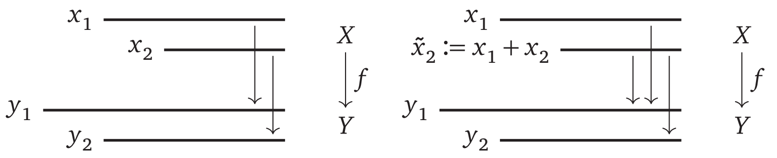
$$\Delta f = \sum_{b_i = d_j} f_{ij}, \quad \Delta g = \sum_{d_j = t_k} g_{jk}, \quad \text{and} \quad \Delta(g \circ f) = \sum_{b_i = t_k} (g \circ f)_{ik}.$$

Thus we get

$$(\Delta(g \circ f))_{ik} = \sum_{\substack{j \\ b_i \geq d_j \geq t_k \\ b_i = t_k}} g_{jk} \circ f_{ij} = \sum_{\substack{j \\ b_i = d_j = t_k}} g_{jk} \circ f_{ij} = (\Delta g \circ \Delta f)_{ik},$$

which shows the compatibility with composition. □

C.30 **Corollary.** *Let $f\colon X \hookrightarrow Y$ be a monomorphism in* **Bar**. *Then*



$$\begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} x_1 & \tilde{x}_2 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

Figure C.5: Two representations of the same morphism $f\colon X \to Y$ in **Pfd**, as bars with arrows and as matrices. On the left, the basis $\{x_1, x_2\}$ makes the monomorphism strictly bar-to-bar. This is not the case for the basis $\{x_1, \tilde{x}_2\}$ chosen on the right.

1. *If f is bar-to-bar, then it is isomorphic to Δf.*
2. *If f is strictly bar-to-bar, then $f = \Delta f$.*

Saying that two morphisms $f, g\colon X \to Y$ are **isomorphic** means that there exist automorphisms $\varphi\colon X \to X$ and $\psi\colon Y \to Y$ such that $f = \psi \circ g \circ \varphi$.

Proof. If f is a strictly bar-to-bar monomorphism, then it is of the form

$$\bigoplus_i (K(a_i, b_i) \hookrightarrow K(c_i, b_i)) \oplus \bigoplus_j (0 \to K(c_j, d_j))$$

where $X = \bigoplus_i K(a_i, b_i)$ and $Y = \bigoplus_i K(c_i, b_i) \oplus \bigoplus_j K(c_j, d_j)$. The second statement then follows from the definition of Δ.

For the first statement, by definition, there exist automorphisms $\varphi_X\colon X \to X$ and $\varphi_Y\colon Y \to Y$ such that $\varphi_Y \circ f \circ \varphi_X^{-1}$ is strictly bar-to-bar. Then

$$\varphi_Y \circ f \circ \varphi_X^{-1} = \Delta(\varphi_Y \circ f \circ \varphi_X^{-1}) = \Delta\varphi_Y \circ \Delta f \circ (\Delta\varphi_X)^{-1}.$$

Since Δ is a functor, it sends isomorphisms to isomorphisms. Thus Δf is isomorphic to f. □

C.31 **Remark.** The converse of the first statement is also true, but it follows from Theorem C.35. The converse of the second statement is not true: see the right side of Figure C.5, where $f = \Delta f$ but f is not strictly bar-to-bar. ■

C.32 **Proposition.** *Let $f\colon X \hookrightarrow Y$ be a monomorphism in* **Bar***. Then Δf is also a monomorphism.*

Proof. Write $X = \bigoplus_i K(a_i, b_i)$ and $Y = \bigoplus_j K(c_j, d_j)$. Recall that Δf is a direct sum of morphisms $\bigoplus_{b_i=b} K(a_i, b_i) \to \bigoplus_{d_j=b} K(c_j, d_j)$ for all $b \in \mathbb{E}$. Thus it suffices to prove, for all $b \in \mathbb{E}$, that Δf restricted to $X^{(b)} := \bigoplus_{b_i=b} K(a_i, b_i)$ is a monomorphism.

Fix $b \in \mathbb{E}$ and write $g := \Delta f|_{X^{(b)}}$. There exists $t \in \mathbb{R}$ such that $t^- < b$ and every bar $K(a_i, b)$ in X satisfies $a_i \leq t^-$. If there were an infinite number of bars $K(c_j, d_j)$ in Y such that $t \in \langle c_j, d_j \rangle$, then Y_t would be infinite-dimensional. Since Y is in **Pfd**, this is not possible, so we conclude that there are only finitely many such bars, and therefore there exists $u \geq t$ in $\mathbb{R}$ such that f maps bars $K(a_i, b_i)$ in X with $b_i = b$ exclusively to bars $K(c_j, d_j)$ in Y with $d_j = b$. Thus, at u, the same-endpoint operator does not affect $f|_{X^{(b)}}$: in other words, $g_u = (f|_{X^{(b)}})_u$. In particular, $g_u = (f|_{X^{(b)}})_u$ is injective.

Next, we check that g_s is injective for all $s < u$. Indeed, if g_s sent a nonzero element $x \in X_s^{(b)}$ to $0 \in Y_s$, then by the transition maps we would get $X_{s \leq u}(x) \neq 0$ and $g_t(X_{s \leq u}(x)) = 0$, contradicting the injectivity of g_u. For $s \in \langle u^+, b \rangle$, injectivity holds because $g_s = (f|_{X^{(b)}})_s$ by the same argument as before. Thus g is pointwise injective, so it is a monomorphism. This concludes the proof. □

C.33 **Proposition.** *Let $f\colon X \to Y$ be a morphism in* **Pfd***. Let $\varphi_X\colon X \cong \bigoplus_i K(a_i, b_i)$ and $\varphi_Y\colon Y \cong \bigoplus_j K(c_j, d_j)$ be barcode decompositions of X and Y. Define $\Delta f := \varphi_Y^{-1} \Delta(\varphi_Y f \varphi_X^{-1}) \varphi_X$, where we identify the morphism $\varphi_Y f \varphi_X^{-1}$ in* **Pfd** *with the one in* **Bar***. Then, up to isomorphism, Δf does not depend on the choice of φ_X and φ_Y.*

Proof. Let ψ_X and ψ_Y be other choices of isomorphisms. Then

$$\begin{aligned}
\psi_Y^{-1} \Delta(\psi_Y f \psi_X^{-1}) \psi_X &= \psi_Y^{-1} \Delta(\psi_Y \varphi_Y^{-1} \varphi_Y f \varphi_X^{-1} \varphi_X \psi_X^{-1}) \psi_X \\
&\cong \Delta(\underbrace{\psi_Y \varphi_Y^{-1}}_{\in \mathbf{Bar}} \underbrace{\varphi_Y f \varphi_X^{-1}}_{\in \mathbf{Bar}} \underbrace{\varphi_X \psi_X^{-1}}_{\in \mathbf{Bar}}) \\
&= \Delta(\psi_Y \varphi_Y^{-1}) \Delta(\varphi_Y f \varphi_X^{-1}) \Delta(\varphi_X \psi_X^{-1}) \\
&\qquad\qquad\qquad\qquad \text{(by functoriality in } \mathbf{Bar}\text{)} \\
&\cong \Delta(\varphi_Y f \varphi_X^{-1})
\end{aligned}$$

$$\cong \varphi_Y^{-1}\Delta(\varphi_Y f \varphi_X^{-1})\varphi_X. \qquad \square$$

C.34 **Remark.** In particular, this means that we can define the same-endpoint operator Δ on **Pfd** up to isomorphism. ■

C.35 **Theorem.** *For all morphisms f in* **Pfd**, *the morphism Δf is bar-to-bar.*

C.36 **Remark.** In **Pfd**, the morphism Δf is defined only up to isomorphism, see Proposition C.33. Since the property of being bar-to-bar is an invariant of isomorphism classes of morphisms (by definition), Theorem C.35 is well-formulated. ■

To show Theorem C.35, we will be working with infinite (unlabeled) matrices. In particular with infinite column-finite matrices (see Definition C.11).

In [20] the author presents a reduction algorithm for infinite row-finite matrices (defined analogously to column-finite ones). We proceed similarly with Algorithm C.1: compare with the standard algorithm of persistent homology, see [10].

C.37 **Lemma.** *Algorithm C.1 is well-specified: it takes as input an infinite column-finite matrix M and returns a factorization $A = UMV$ where U and V are upper triangular invertible and A has at most one nonzero entry in each row and column.*

Proof. We observe that after the j^{th} step, the first j columns of V are fixed (they will not be modified in subsequent iterations). To show that the limit matrix V is invertible, we construct its left inverse iteratively, starting with $V^{-1} = I$ and then applying $V^{-1} \leftarrow (I - \lambda E_{j',j})V^{-1}$ every time we set $V \leftarrow V(I - \lambda E_{j',j})$. After the j^{th} step, the row transformations on V^{-1} involve adding rows with index $k > j$ to other rows: this will never affect the first j rows, as V^{-1} is always upper triangular.

For U, the argument is a little subtler. For all j, there exists a positive integer J such that, in the fully reduced matrix A, for all $j' > J$, $\operatorname{low}(j') > \operatorname{low}(j)$. Thus, after the J^{th} step, the j^{th} column of U is no longer affected by the row transformations that update U. To show that the limit matrix U is invertible, we construct its right inverse iteratively, starting with $U^{-1} = I$ and then applying

Algorithm C.1: Infinite column-finite (unlabeled) matrix reduction. I denotes the infinite identity matrix, $E_{i,j}$ denotes the infinite matrix with all 0's except a 1 in position (i,j), and $\mathrm{low}(j) := \max\{i \geq 1 \mid A_{i,j} \neq 0\}$.

Input: an infinite column-finite matrix M.

Output: a factorization $A = UMV$ where U and V are upper triangular invertible and A has at most one nonzero entry in each row and column.

$U \leftarrow I$;
$V \leftarrow I$;
$A \leftarrow M$;
for $j = 1, 2, \ldots$ **do**
  while $\exists j' < j$, $\mathrm{low}(j') = \mathrm{low}(j)$ **do** `// reduce by columns with same low`
    $\lambda \leftarrow A_{\mathrm{low}(j),j} / A_{\mathrm{low}(j'),j'}$;
    $A_{*,j} \leftarrow A_{*,j} - \lambda A_{*,j'}$; `// equivalently,` $A \leftarrow A(I - \lambda E_{j',j})$
    $V \leftarrow V(I - \lambda E_{j',j})$;
  end
  $i \leftarrow \mathrm{low}(j)$;
  while $\exists i' < i$, $A_{i',j} \neq 0$ **do** `// reduce nonzero entries above low`
    $\lambda \leftarrow A_{i',j} / A_{i,j}$;
    $A_{i',*} \leftarrow A_{i',*} - \lambda A_{i,*}$; `// equivalently,` $A \leftarrow (I - \lambda E_{i',i})A$
    $U \leftarrow (I - \lambda E_{i',i})U$;
  end
end
return A, U, V

$U^{-1} \leftarrow U^{-1}(I - \lambda E_{i',i})$ every time we set $U \leftarrow (I - \lambda E_{i',i})U$. After the J^{th} step, the only columns that get transformed are those with index $> \text{low}(j)$, so in particular the j^{th} column is no longer affected by the column transformations that update U^{-1}.

Finally, for A, we use the same arguments as for V and U to conclude that A converges to a well-defined limit matrix. □

Proof of Theorem C.35. It suffices to prove the result for f in **Bar**, so we assume this. In this case, the morphism $\Delta f\colon X \to Y$ is a direct sum of morphisms between the direct summands of X and Y where all bars share a common end-point. Δf is bar-to-bar if each summand is bar-to-bar, so, without loss of generality, we can assume that X and Y are persistence modules where all bars share a common end-point. Following Remark C.8, we can then represent Δf as an infinite matrix M with countably many columns and rows, and where the columns and rows correspond to bars and are sorted by start-point.

We observe that every column of M has a finite number of nonzero entries. Indeed, if the column corresponding to a bar $K(a,b)$ of X had an infinite number of nonzero entries, then $K(a,b)$ would map to an infinite sequence $(K(c_i,b))_i$ of bars of Y, with $c_i \leq a$ for all i. Then (since we do not consider empty bars), there exists a point $t \in \langle a,b\rangle \subseteq \langle c_i,b\rangle$ for all i, and so Y_t is infinite-dimensional, which contradicts the definition of **Bar**.

Thus we can apply Algorithm C.1 and Lemma C.37 to obtain invertible upper triangular matrices U and V such that UMV has at most one nonzero entry in each row and column: this corresponds to a strictly bar-to-bar morphism from X to Y. Since U and V are upper triangular, they correspond to automorphisms of X and Y, respectively. Therefore the morphism f is isomorphic to a strictly bar-to-bar morphism. In other words, f is bar-to-bar. □

C.3.3 Induced matchings

Recall that we use the the category **Barc** from [4]. While for their canonically induced matchings compose naturally, we need more structure on our persistence modules to make the matchings compose in an intuitive way.

C.38 **Definition** (Matching operators). A **matching operator** is a function μ that takes a morphism $f\colon X \to Y$ in **Bar** and returns a matching (i.e., morphism)

$\mu(f)\colon \mathscr{B}(X) \to \mathscr{B}(Y)$ in **Barc**. The second argument is sometimes omitted. The operator μ is

1. **additive** if, given morphisms f and g, $\mu(f \oplus g) = \mu(f) \sqcup \mu(g)$.
2. **split** if, for a strictly bar-to-bar morphism f, $\mu(f)$ is the matching of Definition C.21[2].
3. **functorial on monomorphisms** (resp. **epimorphisms**) if, given composable monomorphisms (resp. epimorphisms) f and g, $\mu(g \circ f) = \mu(g) \bullet \mu(f)$. ■

C.39 **Remark.** 1. Matching operators are well-defined. In particular, they are equivariant under reindexing of the barcode multiset representations [4].

2. By [3, Proposition 5.10], such operators are never functorial over all morphisms. Moreover, the **canonically induced matching** of that paper is functorial on monomorphisms and epimorphisms.
3. Note that being split is equivalent to being additive and mapping morphisms $K(a,b) \to K(c,d)$ between single bars nontrivially.
4. Using the axiom of choice, and viewing morphisms as objects in the additive category $\mathrm{Fun}(\mathbf{R} \times \{0 < 1\}, \mathbf{Vect}_K)$, we can define arbitrary additive matchings by choosing a matching for each indecomposable morphism. ■

We now show that being split and functorial on monomorphisms (resp. epipmorphisms) are mutually exclusive.

C.40 **Proposition.** *A matching operator cannot be split and functorial on monomorphisms (resp. epimorphisms).*

Proof. We proceed by contradiction: suppose that such an operator μ exists. Let $b > a_1 > a_2 > \cdots > a_6$ be elements of $\mathbb{E}$, and consider the sequence

$$X = K(a_1, b) \oplus K(a_2, b) \overset{f}{\hookrightarrow} Y = K(a_3, b) \oplus K(a_4, b) \overset{g}{\hookrightarrow} Z = K(a_5, b) \oplus K(a_6, b)$$

[2] more precisely, the definition's matching is $\{(i,j) \mid ((\langle a_i, b_i \rangle, i), (\langle c_j, d_j \rangle, j)) \in \mu(f)\}$.

where the two monomorphisms are represented by the labeled matrices

$$f = \begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_4 \\ a_3 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad g = \begin{array}{c c} & \begin{array}{cc} a_4 & a_3 \end{array} \\ \begin{array}{c} a_6 \\ a_5 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}, \quad \text{and so} \quad gf = \begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_6 \\ a_5 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array},$$

where the label a_i corresponds to the interval $\langle a_i, b\rangle$. Recall that isomorphisms of morphisms correspond to certain left-to-right column transformations and bottom-to-top row transformations of their matrix representations. Adding the left column to the right column in the matrix representation of gf, we obtain the isomorphism

$$gf \cong \begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_6 \\ a_5 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} = \underbrace{\begin{array}{c c} & \begin{array}{cc} a_4 & a_3 \end{array} \\ \begin{array}{c} a_6 \\ a_5 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}}_{=: h_0} \circ \begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_4 \\ a_3 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}. \tag{C.6}$$

In other words, gf and $h_0 f$ are isomorphic. Moreover, we can decompose f as

$$\begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_4 \\ a_3 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} = \underbrace{\begin{array}{c c} & \begin{array}{cc} a_4 & a_3 \end{array} \\ \begin{array}{c} a_4 \\ a_3 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array}}_{=: h_1} \circ \underbrace{\begin{array}{c c} & \begin{array}{cc} a_2 & a_1 \end{array} \\ \begin{array}{c} a_4 \\ a_3 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}}_{=: h_2}$$

The morphism h_1 is an isomorphism, and by splitness $\mu(h_2)$ is also an isomorphism, so by functoriality (recall also that, by functoriality, μ sends isomorphisms to isomorphisms), we deduce that $\mu(f) = \mu(h_1) \circ \mu(h_2)$ is an isomorphism. Applying μ to (C.6), by functoriality, we get $\mu(g) \circ \mu(f) \cong \mu(h_0) \circ \mu(f)$. As we just showed that $\mu(f)$ is invertible, we deduce that $\mu(g) \cong \mu(h_0)$. But the only automorphisms of $\mathscr{B}(Y)$ and $\mathscr{B}(Z)$ are the identity matchings, so $\mu(g) = \mu(h_0)$. But this contradicts the assumption that μ is split, since g and h_0 are different strictly bar-to-bar morphisms. We conclude that there is no induced matching representation operator that is both split and functorial on monomorphisms. The epimorphism case holds by a dual argument. $\square$

We can now define our induced matching.

C.41 **Definition.** Given a morphism $f\colon X \to Y$ in **Bar**, we know by Theorem C.35 that Δf is bar-to-bar. Specifically, Algorithm C.1 induces an overlap matching $\mu_\Delta(f) \in \mathbf{Barc}(\mathscr{B}(X), \mathscr{B}(Y))$. ■

C.42 **Remark.**

1. In the case where X and Y have finitely many bars, this induced matching is constructed for monomorphisms and epimorphisms in [6, Theorems 6.11 & 6.13], where it is called an *induced algebraic matching*.
2. We can show that the matching operator μ_Δ is additive. Moreover, it is split by construction. As a consequence, it is not functorial on either monomorphisms nor epimorphisms.
3. The matching operator μ_Δ is not very informative on arbitrary morphisms. Indeed, it is empty as soon as no bars of X or Y have the same end-point.

■

Instead, we may follow the strategy of [3]: given a morphism $f\colon X \to Y$ in **Bar**, consider an epi-mono factorization of f,

$$X \xrightarrow{e} \operatorname{im}(f) \xrightarrow{m} Y,$$

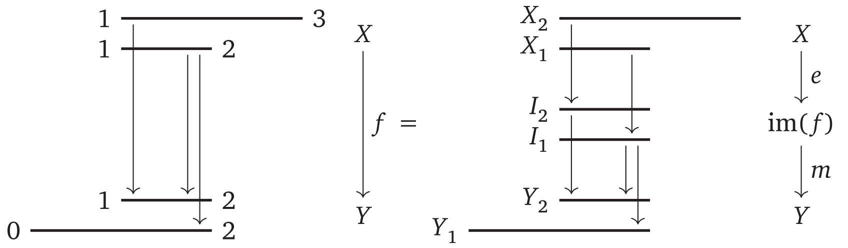
where we define $\operatorname{im}(f)$ as the object in **Bar** that isomorphic to the image of f in **Pfd**, i.e. it is the sum of the bars in the barcode of the image. We would then define the **matching induced by** f to be the matching

$$\mu(f) \coloneqq \mu_\Delta(m) \bullet \mu_\Delta(e^*)^* \tag{C.7}$$

in **Barc**, where $-^*$ is the dual operator for epimorphisms and matchings (see Section C.2.4).

The issue with this construction is that the epi-mono factorization is only unique up to isomorphism: there exist other factorizations, which are all of the form $(\varphi^{-1} \circ e, m \circ \varphi)$ with φ an automorphism on $\operatorname{im}(f)$. Indeed, we have the following example, which shows that the induced matching depends on the choice of factorization.

C.43 **Example.** Consider the following morphism (over the field $\mathbb{F}_2$) along with an epi-mono factorization:



where, on the right, we have named all of the bars. The image $\operatorname{im}(f)$ consists of two bars I_1 and I_2 with identical support $\langle 1,2\rangle$. We have the matrix representations

$$F = \begin{array}{c c} & \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}, \quad E = \begin{array}{c c} & \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array},$$

$$\text{and} \quad M = \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \cong \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

for f, e, and m, respectively (recall that we are working over the field $\mathbb{F}_2$). The induced matching is

$$\begin{aligned} \mu_\Delta(m) \bullet \mu_\Delta(e^*)^* &= \{(I_1,Y_2),(I_2,Y_1)\} \bullet \{(X_1,I_1),(X_2,I_2)\} \\ &= \{(X_1,Y_2),(X_2,Y_1)\}. \end{aligned}$$

Now consider $\varphi\colon \operatorname{im}(f) \to \operatorname{im}(f)$ the automorphism that permutes the two bars, with matrix representation

$$\Phi = \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}.$$

Since $\Phi^{-1} = \Phi$, we compute

$$\Phi^{-1}E = \begin{array}{c c} & \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{array}{c} I_1 \\ I_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M\Phi = \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \cong \begin{array}{c c} & \begin{array}{cc} I_1 & I_2 \end{array} \\ \begin{array}{c} Y_1 \\ Y_2 \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

and find the induced matching

$$\begin{aligned}\mu_\Delta(m \circ \varphi) \bullet \mu_\Delta((\varphi^{-1} \circ e)^*)^* &= \{(I_1, Y_2), (I_2, Y_1)\} \bullet \{(X_1, I_2), (X_2, I_1)\} \\ &= \{(X_1, Y_1), (X_2, Y_2)\} \\ &\neq \mu_\Delta(m) \bullet \mu_\Delta(e^*)^*.\end{aligned}$$

Thus the induced matching as defined in Equation (C.7) is not unique. ■

Induced matchings and automorphisms of persistence modules interact as follows.

C.44 **Proposition.** *In the category* **Bar**, *let $f\colon X \to Y$ be a morphism and $\varphi\colon X \to X$ and $\psi\colon Y \to Y$ automorphisms. Then there exist automorphic matchings $\sigma\colon \mathscr{B}(X) \to \mathscr{B}(X)$ and $\tau\colon \mathscr{B}(Y) \to \mathscr{B}(Y)$ such that*

$$\mu_\Delta(\psi \circ f \circ \varphi) = \tau \bullet \mu_\Delta(f) \bullet \sigma.$$

Proof. The morphisms f and $\psi \circ f \circ \varphi$ are isomorphic to each other. By Proposition C.33, $\Delta(f)$ and $\Delta(\psi \circ f \circ \varphi)$ are isomorphic. By definition of μ_Δ and the arguments of the proof of Proposition C.24 (the induced matching is read from the decomposition of f as a representation of $\mathbf{R} \times \{0 < 1\}$), the induced matchings $\mu_\Delta(f)$ and $\mu_\Delta(\psi \circ f \circ \varphi)$ are equal up to permutations of bars with identical support. By the overlap condition for composition in **Barc**, automorphic matchings are exactly the bijective matchings that, at most, only permute of bars with identical support. We conclude that there exist automorphic matchings $\sigma\colon \mathscr{B}(X) \to \mathscr{B}(X)$ and $\tau\colon \mathscr{B}(Y) \to \mathscr{B}(Y)$ such that

$$\mu_\Delta(\psi \circ f \circ \varphi) = \tau \bullet \mu_\Delta(f) \bullet \sigma. \qquad \square$$

The automorphic matchings σ and τ depend on the morphism f, as shown by the following example.

C.45 **Example.** Let f and $g\colon X \to Y$ be morphisms in **Bar** represented by the matrices

$$F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

respectively, where the two bars of X have the same support, and the two bars of Y have the same support (but not necessarily the same as that of X). Let φ

be the automorphism on X represented by

$$\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$F\Phi = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad G\Phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Reading the pivots of the relevant matrices, we find that $\mu_\Delta(f) = \mu_\Delta(f\varphi)$ but $\mu_\Delta(g) \neq \mu_\Delta(g\varphi)$. In other words, the automorphic matching σ induced by φ via Proposition C.44 depends on the morphism, with which it is composed. ■

C.3.4 Example: recovering the $\mathscr{P}$-matching

We can define an induced matching on **Pfd** as soon as we determine a strategy for selecting, for a given morphism f in **Pfd**, an epi-mono factorization $f = me$. As an example, in this section we recover the induced matchings of [15, 16] via epi-mono factorizations.

In this section we assume that our persistence modules have countably many bars. For example, this happens if we assume that they have no ephemeral bars.

We provide a simpler method of computing the image of a morphism in **Bar**; compare with [24].

C.46 **Proposition.** *Let $f\colon X \to Y$ be a morphism in* **Bar** *and write $X = \bigoplus_{j\in J} K(a_j, b_j)$ and $Y = \bigoplus_{i\in I} K(c_i, d_i)$. We define the strictly bar-to-bar epimorphism*

$$\tilde{X} := \bigoplus_{j\in J} K(a_j, \infty) \twoheadrightarrow X$$

and the strictly bar-to-bar monomorphism

$$Y \hookrightarrow \tilde{Y} := \bigoplus_{i\in I} K(-\infty, d_i).$$

This gives us the morphism

$$\tilde{f} := \tilde{X} \twoheadrightarrow X \xrightarrow{f} Y \hookrightarrow \tilde{Y}.$$

Let $\tilde{F}$ be an ordered matrix representation of $\tilde{f}$. Then we can apply Algorithm C.2 to reduce the matrix $\tilde{F}$ to a matrix A with at most one nonzero

Algorithm C.2: Infinite column-finite labeled matrix reduction. Notations are as in Algorithm C.1.

Input: an infinite column-finite labeled matrix M where the sequence of row labels and the sequence of column labels are totally ordered.

Output: a factorization $A = UMV$ where U and V are upper triangular invertible and A has at most one nonzero entry in each row and column.

$U \leftarrow I$;
$V \leftarrow I$;
$A \leftarrow M$;
for $j = 1, 2, \ldots$ **do**
  while $\exists j' < j$, $\mathrm{low}(j') = \mathrm{low}(j)$ **do** `// reduce by columns with same low`
    $\lambda \leftarrow A_{\mathrm{low}(j),j} / A_{\mathrm{low}(j'),j'}$;
    $A \leftarrow A(I - \lambda E_{j',j})$;
    $V \leftarrow V(I - \lambda E_{j',j})$;
  end
  $i \leftarrow \mathrm{low}(j)$;
  while $\exists i' < i$, $A_{i',j} \neq 0$ **do** `// reduce nonzero entries above low`
    $\lambda \leftarrow A_{i',j} / A_{i,j}$;
    $A \leftarrow (I - \lambda E_{i',i})A$;
    $U \leftarrow (I - \lambda E_{i',i})U$;
  end
end
return A, U, V

coefficient in each row and column. Moreover, these pivots of A give the barcode of $\mathrm{im}(f)$*:*

$$\mathscr{B}(\mathrm{im}(f)) = \{\langle a_j, d_i \rangle \mid A_{ij} \neq 0\}.$$

Proof. First, we observe that $\mathrm{im}(f) \cong \mathrm{im}(\tilde{f})$. Next, we recall that we have assumed that X and Y have countably many bars. Moreover, the row labels (resp. column labels) of $\tilde{F}$ are totally ordered, as they all share the same start-point (resp. end-point). We can therefore apply Algorithm C.2 with $\tilde{F}$ as input. By the assumption that the row and column labels are totally ordered, we know that all left-to-right column operations and bottom-to-top row operations on the matrix $\tilde{F}$ are valid matrix operations.

Note that Algorithm C.2 is very similar to Algorithm C.1, the difference being that we are reducing a labeled matrix instead of an unlabeled matrix. In particular, we use matrix multiplications, specifically overlap multiplications (Definition C.12), to denote the column and row operations. The proof of correctness of the algorithm is analogous to Lemma C.37. Thus we can apply Algorithm C.2 to obtain a matrix A. As in Lemma C.37, this matrix has at most one nonzero coefficient in each row and column. In particular, we obtain the barcode of the image of the morphism that the matrix A represents: explicitly, this barcode is

$$\{\langle a_j, d_i\rangle \mid A_{ij} \neq 0\}.$$

Moreover, since A is a reduction of $\tilde{F}$, the image of A is isomorphic to $\mathrm{im}(\tilde{f}) \cong \mathrm{im}(f)$. This concludes the proof. □

C.47 **Remark.** The morphism $\tilde{f}\colon \tilde{X} \to \tilde{Y}$ is a **fringe presentation** of $\mathrm{im}(f)$ as defined in [19]. We also note that the ordered matrix representation of $\tilde{F}$ is exactly that of F, but with relabeled rows and columns. Thus Proposition C.46 provides an effective way of computing the barcode of the image of a morphism, provided a matrix representation of the morphism. ■

Let $f\colon X \to Y$ be a morphism in **Bar**. Denote by F the matrix representation of f where rows are ordered (lexicographically) by end-point then start-point and the columns by start-point then end-point. Applying Proposition C.46, we obtain a matrix $\tilde{F}$ with the same coefficients as F but different labels, and $A = U\tilde{F}V$ its reduced form, with U and V invertible upper triangular matrices.

Let $R := U^{-1}A$ and $\tilde{I}$ be the subset of indices in I such that row i of A (and R) contains a pivot. Let E be a matrix whose columns are labeled like F and whose rows are indexed by I. We label row i of E by the interval $\langle a_j, d_i\rangle$, corresponding to the pivot (i,j) of A. We set

$$E_{ij} := \begin{cases} R_{ij} & \text{if } i = \mathrm{low}(j) \text{ in } R,\\ 0 & \text{otherwise.}\end{cases}$$

C.48 **Proposition.** *We use the notations introduced immediately above. Let Z be the persistence module in* **Bar** *that is isomorphic to* $\mathrm{im}(f)$. *There exists an epi-mono factorization $f = m \circ e$ where the epimorphism $e\colon X \to Z$ is represented* *by the matrix EV^{-1} and the monomorphism $m\colon Z \to Y$ is represented by a matrix M in row echelon form with pivots $\{(i,i) \mid i \in \tilde{I}\}$.*

Proof. We see that EV^{-1} represents an epimorphism from X to $\mathrm{im}(f)$.

Let us solve the equation $ME = R := U^{-1}A$ for the matrix M. Note that the labeling of M is determined by this equation. By construction, E has at most one nonzero coefficient in each column. Therefore, a given unknown $M_{i,i'}$ in M appears in at most one equation:

- If there exists $j \in J$ such that $i' = \mathrm{low}(j)$ in R, then $E_{i',j}$ is nonzero and $M_{i,i'} = M_{i,i'}$ must satisfy the equation
$$M_{i,i'}E_{i',j} = R_{ij}.$$
If $R_{ij} = 0$, then we set $M_{i,i'} = 0$. Otherwise, we need to check that we can set $M_{i,i'}$ to be nonzero. We know that $i \leq i'$ ($R_{i',j}$ is the pivot, so R_{ij} must be above it). Column i' of M is labeled by $\langle a_j, d_{i'}\rangle$ and row i of M is labeled by $\langle c_i, d_i\rangle$. We have $d_{i'} \geq d_i$ because $i' \geq i$ and we assume that the $\langle c_i, d_i\rangle$ are ordered compatibly with the product order. We also have $a_j \geq c_i$ because R_{ij} is nonzero (and so $\langle a_j, b_j\rangle$ must overlap $\langle c_i, d_i\rangle$ above). Thus we have checked that $\langle a_j, d_{i'}\rangle$ overlaps $\langle c_i, d_i\rangle$ above, so we can set $M_{i,i'}$ to be nonzero.
- Otherwise, $M_{i,i'}$ has no equations to satisfy, and we choose to set $M_{i,i'} = 0$.

We observe that the resulting M is in reduced row echelon form, with pivots at the coordinates (i,i) for i in $\tilde{I}$. Each column as a pivot and the pivots' row and column labels have the same end-point d_i, so we deduce that M represents a monomorphism: indeed, at every time $t \in \mathbf{R}$, M_t represents an injective linear map.

We thus obtain $F = MEV^{-1}$, giving us an epi-mono factorization $f = me$ by the morphisms e and m represented by EV^{-1} and M, respectively. □

C.49 **Definition.** We define the induced matching $\mu(f) := \mu_\Delta(m) \bullet \mu_\Delta(e^*)^*$. ■

C.50 **Remark.** By the last observation on M, we deduce that the induced matching $\mu_\Delta(m)$ matches each bar $\langle a_j, d_i\rangle$ in the image to $\langle c_i, d_i\rangle$ in $\mathscr{B}(Y)$. In other words, we know the matching $\mu(f)$ as soon as we know $\mu_\Delta(e^*)^*$, or as soon as we know the reduced matrix A. ■

In the case where X and Y are tame persistence modules (i.e., they have finite barcodes), [16, Algorithm 1] is a matrix reduction algorithm to obtain a multiset matching from a morphism $f\colon X \to Y$, which we will call the $\mathscr{P}$-matching.

We first show the pairing uniqueness lemma [10] for tame fringe presentations, following the reasoning and notations of the original proof. Given a matrix $A \in K^{m\times n}$, for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, denote by A_i^j the lower left minor of A obtained by removing the first $i-1$ rows and the last $n-j$ columns. Define

$$r_A(i,j) := \operatorname{rank} A_i^j - \operatorname{rank} A_{i+1}^j - \operatorname{rank} A_i^{j-1} + \operatorname{rank} A_{i+1}^{j-1},$$

where $\operatorname{rank} A$ denotes the number of bars in the image of the morphism represented by A.

C.51 **Lemma** (Pairing uniqueness lemma). *Let $\tilde{F} \in K^{m\times n}$ be the relabeled matrix representation of a morphism $f\colon X \to Y$ as in Proposition C.46. Now consider the reduced matrix A returned by Algorithm C.2. Then $A_{i,j} \neq 0$ if, and only if, $r_{\tilde{F}}(i,j) = 1$.*

Proof. Note that left-to-right column operations and bottom-to-top row operations correspond to isomorphism actions on the domain and codomain of the represented morphism. This holds for the restricted matrices A_i^j, and therefore $r_{\tilde{F}} = r_A$. It therefore suffices to show that $A_{i,j} \neq 0$ if, and only if, $r_A(i,j) = 1$, and this follows immediately from the fact that A has at most one nonzero coefficient in each row and column. □

C.52 **Proposition.** *Let $\tilde{F}$ be the relabeled matrix representation of a morphism $f\colon X \to Y$ as in Proposition C.46. Then the reduction of the matrix $\tilde{F}$ by Algorithm C.2 and the reduction of the matrix F by [16, Algorithm 1] have the same pivots.*

Proof. By Lemma C.51, the order in which we do column and row operations in Algorithm C.2 does not affect the resulting pivots. Moreover, we note that the row operations do not affect the pivots of the matrix, and therefore we will ignore that part of Algorithm C.2. We can therefore assume that we do column reductions in the same order as in [16, Algorithm 1]: that is, start from the columns with the lowest nonzero coefficient and reduce them before moving to the columns with the next lowest nonzero coefficient, and so on.

The only difference between the two algorithms is then how invalid matrix coefficients are treated. In Algorithm C.2, a coefficient is set to zero if the column label does not overlap above the row label; in our case where the column labels all end at $+\infty$ and the row labels start at $-\infty$, this only happens when the column and row labels are disjoint. In [16, Algorithm 1], a coefficient is set to zero if the column and row labels in F are disjoint. This happens only in the following situation. There are two columns j and j' with $j' < j$ such that $\mathrm{low}(j) = \mathrm{low}(j') =: i$. When adding column j' to column $j > j'$, a nonzero coefficient on row $i' > i$ produces an invalid nonzero coefficient at (i', j). Denoting by $[a_j, b_j)$, $[a_{j'}, b_{j'})$, $[c_i, d_i)$, and $[c_{i'}, d_{i'})$ the labels of the relevant columns and rows, we have the following inequalities. By assumption on the order of the rows and columns, we have $a_{j'} \leq a_j$ and $d_{i'} < d_i$. By the validity of the nonzero coefficient at (i', j'), we have $c_{i'} \leq a_{j'} < d_{i'} \leq b_{j'}$. By the validity of the nonzero coefficient at (i, j), we have $c_i \leq a_j < d_i \leq b_j$. Thus we have $c_{i'} < a_j$ and $d_{i'} < b_j$. In order for the coefficient at (i', j) to be invalid, it is therefore necessary that $a_j < d_{i'}$ does not hold, i.e., that $a_j \geq d_{i'}$. We conclude that the same coefficient (i', j) in $\tilde{F}$ would have disjoint row and column labels, and therefore would be canceled by the overlap multiplication.

We conclude that Algorithm C.2 and [16, Algorithm 1] return matrices with the same pivots. □

We deduce that our induced matching $\mu(f)$ is a representation of the (multiset) $\mathscr{P}$-matching defined in [16].

C.4 Application to Wasserstein distances

C.4.1 Sizes and ranks of (co)kernels

In this section we prove some results in the category **Pfd**. In particular, we highlight a connection between rank functions and p-norms, which yields new proofs of results in [1, Section 3]. We denote by $\mathrm{rk} f$ the rank of a linear map f.

C.53 **Proposition.** *Let $f\colon X \hookrightarrow Y$ be a monomorphism in* **Pfd** *and consider Δf the functorially defined bar-to-bar morphism and f_φ the bar-to-bar monomorphism induced by the canonical matching between X and Y (this assumes that we fix barcodes). Then, for all $s \leq t$ in $[0, \infty)$,*

$$\mathrm{rk}((\mathrm{coker}\, f)_{s \leq t}) \geq \mathrm{rk}((\mathrm{coker}\, \Delta f)_{s \leq t}) \geq \mathrm{rk}((\mathrm{coker}\, f_\varphi)_{s \leq t}).$$

We first prove a series of lemmas.

C.54 **Lemma.** *Let $f\colon X \hookrightarrow Y$ be an arbitrary monomorphism in* **Pfd**. *For all $s \le t$ in $[0,\infty)$,*

$$\operatorname{rk}((\operatorname{coker} f)_{s\le t}) = \operatorname{rk}(Y_{s\le t} + f_t) - \dim X_t,$$

where $Y_{s\le t} + f_t\colon Y_s \oplus X_t \to Y_t$.

Proof. Consider the following diagram of vector spaces:

$$\begin{tikzcd}
X_s \arrow[r, hook, "f_s"] \arrow[d, "X_{s\le t}"'] & Y_s \arrow[r, two heads] \arrow[d, "Y_{s\le t}"'] & (\operatorname{coker} f)_s \arrow[d, "(\operatorname{coker} f)_{s\le t}"] \\
X_t \arrow[r, hook, "f_t"'] & Y_t \arrow[r, two heads] \arrow[d, two heads] & (\operatorname{coker} f)_t \arrow[d, two heads] \\
& \operatorname{coker} Y_{s\le t} \arrow[r, two heads] & C
\end{tikzcd}$$

where C is the cokernel of the map $(\operatorname{coker} f)_{s\le t}$. We observe (by a diagram chase, for instance) that C is also the cokernel of the map $g := Y_{s\le t} + f_t\colon Y_s \oplus X_t \to Y_t$. Thus $\dim C = \dim Y_t - \operatorname{rk} g$, and we conclude the proof by observing that

$$\begin{aligned}
\operatorname{rk}((\operatorname{coker} f)_{s\le t}) &= \dim(\operatorname{coker} f)_t - \dim C \\
&= \dim Y_t - \dim X_t - \dim Y_t + \operatorname{rk} g \\
&= \operatorname{rk} g - \dim X_t.
\end{aligned}$$

□

C.55 **Lemma.** *Let $g\colon X \to Y$ be an arbitrary morphism in* **Pfd**. *Then, for all t in $[0,\infty)$,*

$$\operatorname{rk} g_t \ge \operatorname{rk}(\Delta g)_t.$$

Proof. Fix barcode decompositions of X and Y. Then we can represent g_t as a (finite) matrix, where the columns and rows correspond to bars of X and Y that are nonzero at t, and they are sorted such that the death time is increasing. The matrix M of g_t is then block upper triangular, where the blocks correspond to bars with the same death time. The matrix M_Δ of $(\Delta g)_t$ is the block diagonal part of this matrix.

Let us show that the rank of M_Δ is smaller than the rank of M. Indeed, for every block of M_Δ, select a maximal set of linearly independent columns. The corresponding columns in M are also linearly independent, and therefore the linear map sending each of the selected columns of M_Δ to the corresponding column in M is injective. This defines an injective map $\operatorname{im} M_\Delta \hookrightarrow \operatorname{im} M$. Taking the dimension of both sides, we get $\operatorname{rk}(\Delta g)_t \leq \operatorname{rk} g_t$. □

C.56 **Lemma.** *Let $f\colon X \hookrightarrow Y$ be an arbitrary monomorphism in* **Pfd**. *For all $s \leq t$ in $[0,\infty)$,*

$$\operatorname{rk}(Y_{s\leq t} + f_t) \geq \operatorname{rk}(Y_{s\leq t} + (\Delta f)_t),$$

where $Y_{s\leq t} + f_t$ and $Y_{s\leq t} + (\Delta f)_t\colon Y_s \oplus X_t \to Y_t$.

Proof. Since we are concerned with ranks, which are invariant by isomorphisms of morphisms, we can assume without loss of generality that X, Y, and f are in **Bar**.

Consider $Y[-\delta]$, the persistence module defined as $Y[-\delta]_a := Y_{a-\delta}$. Then there exists a natural morphism in **Pfd** $Y_{\bullet-\delta\leq\bullet}\colon Y[-\delta] \to Y$. The image of this morphism then has the same barcode as Y, except that every birth time has been increased by δ, with resulting bars with a birth time greater than the death time being removed. Denote by $\iota\colon \operatorname{im} Y_{\bullet-\delta\leq\bullet} \hookrightarrow Y$ the monomorphism through which $Y_{\bullet-\delta\leq\bullet}$ factors. Then $\operatorname{rk}(Y_{s\leq t} + f_t) = \operatorname{rk}(\iota_t + f_t)$. Moreover, ι is strictly bar-to-bar, so by Corollary C.30 $\Delta\iota = \iota$. By linearity of Δ (Proposition C.28), we deduce that $\Delta(\iota + f) = \iota + \Delta f$. We conclude as follows:

$$\begin{aligned}
\operatorname{rk}(Y_{s\leq t} + f_t) &= \operatorname{rk}(\iota + f)_t \\
&\geq \operatorname{rk}(\Delta(\iota + f))_t && \text{(by Lemma C.55)} \\
&= \operatorname{rk}(\iota_t + (\Delta f)_t) \\
&= \operatorname{rk}(Y_{s\leq t} + (\Delta f)_t). && \square
\end{aligned}$$

C.57 **Lemma.** *Suppose that all of the bars of X and Y have the same end-point and there exists a monomorphism $X \hookrightarrow Y$ in* **Pfd**. *Then*

$$\operatorname{rk}((\operatorname{coker} f_\varphi)_{s\leq t}) = (\operatorname{rk} Y_{s\leq t} - \dim X_t)_+,$$

where $(-)_+$ is the positive part function.

Proof. Denote by b this common end-point. Since f_φ is induced by the canonical matching, we can assume without loss of generality that $X = \bigoplus_{i=1}^m K(a_i, b)$,

$Y = \bigoplus_{i=1}^n K(c_i, b)$ with $m \le n$ in $\mathbf{N} \cup \{\omega\}$, the a_i and c_i are ordered in nondecreasing order, and the canonical matching φ is the inclusion of $\{1, \dots, m\}$ in $\{1, \dots, n\}$. The fact that X and Y have at most countably many bars with end-point b is explained in Remark C.8. Then

$$f_\varphi = \bigoplus_{i=1}^m (K(a_i, b) \to K(c_i, b)) \oplus \bigoplus_{j=m+1}^n (0 \to K(c_i, b)).$$

Therefore

$$\operatorname{coker} f_\varphi = \bigoplus_{i=1}^m K(c_i, a_i) \oplus \bigoplus_{j=m+1}^n K(c_i, b).$$

If $t^- \ge b$, then the result holds trivially. Now consider the case $t^- < b$. Since the a_i and c_i are in nondecreasing order, there exist integers k_a and k_c such that

$$\begin{aligned} &\{i \in \{1, \dots, m\} : a_i \le t^-\} = \{1, \dots, k_a\} \\ \text{and} \quad &\{i \in \{1, \dots, n\} : c_i \le s^-\} = \{1, \dots, k_c\}. \end{aligned}$$

Therefore

$$\begin{aligned} &\{i \in \{1, \dots, n\} : c_i \le s^-\} \smallsetminus \{i \in \{1, \dots, m\} : a_i \le t^-\} \\ &= \begin{cases} \{k_a + 1, \dots, k_c\} & \text{if } k_a < k_c, \\ \varnothing & \text{otherwise,} \end{cases} \end{aligned}$$

and we conclude that

$$\begin{aligned} \operatorname{rk}((\operatorname{coker} f_\varphi)_{s \le t}) &= \big|\{i \in \{1, \dots, m\} : c_i \le s^-, t^- < a_i\} \\ &\qquad \sqcup \{j \in \{m+1, \dots, n\} : c_i \le s^-\}\big| \\ &= \big|\{i \in \{1, \dots, n\} : c_i \le s^-\} \smallsetminus \{i \in \{1, \dots, m\} : a_i \le t^-\}\big| \\ &= (k_c - k_a)_+ \\ &= (\operatorname{rk} Y_{s \le t} - \dim X_t)_+, \end{aligned}$$

where the second equality comes from the equality between the sets, and the fourth equality comes from the fact that $k_a = \dim X_t$ and $k_c = \operatorname{rk} Y_{s \le t}$. □

Proof of Proposition C.53. We show the first inequality as follows:

$$\begin{aligned} \operatorname{rk}((\operatorname{coker} f)_{s \le t}) &= \operatorname{rk}(Y_{s \le t} + f_t) - \dim X_t && \text{(by Lemma C.54)} \\ &\ge \operatorname{rk}(Y_{s \le t} + (\Delta f)_t) - \dim X_t && \text{(by Lemma C.56)} \end{aligned}$$

$$= \operatorname{rk}((\operatorname{coker}\Delta f)_{s\le t}). \qquad \text{(by Lemma C.54)}$$

We show the second inequality as follows. Since Δf is bar-to-bar by Theorem C.35, its cokernel is the direct sum of the cokernels of the maps restricted to bars with a common end-point. Thus, without loss of generality, we can assume that the bars of X and Y all have the same end-point. In this case, we get

$$\begin{aligned}
\operatorname{rk}((\operatorname{coker}\Delta f)_{s\le t}) &= \operatorname{rk}(Y_{s\le t} + (\Delta f)_t) - \dim X_t && \text{(by Lemma C.54)}\\
&\ge \max\{\operatorname{rk} Y_{s\le t}, \operatorname{rk}(\Delta f)_t\} - \dim X_t && \text{(true for general sums of maps)}\\
&= \max\{\operatorname{rk} Y_{s\le t}, \dim X_t\} - \dim X_t && \text{(since } \Delta f \text{ is a monomorphism)}\\
&= (\operatorname{rk} Y_{s\le t} - \dim X_t)_+\\
&= \operatorname{rk}((\operatorname{coker} f_\varphi)_{s\le t}). && \text{(by Lemma C.57)} \quad \square
\end{aligned}$$

To connect rank functions to p-norms of cokernels, we define the following measurable space.

C.58 **Definition.** Fix $p \in (1,\infty)$. We define the space $\Omega := \{(s,t) : s \le t \in \mathbf{R}\} \subset \mathbf{R}^2$ and the density function $g_p : \mathbf{R}^2 \to [0,\infty)$ given by

$$g_p(s,t) := p(p-1)\,|t-s|^{p-2}. \qquad \blacksquare$$

C.59 **Proposition.** *For X a persistence module and $p \in (1,\infty)$,*

$$\int_\Omega \operatorname{rk} X_{s\le t}\, g_p(s,t)\,\mathrm{d}s\mathrm{d}t = \|X\|_p^p.$$

Proof. We first consider the case where X is a single bar $K(a,b)$. Suppose that a and b are finite (i.e., not $-\infty$ nor $+\infty$, see Definition C.1). Then, for all $p \in (1,\infty)$, we compute

$$\begin{aligned}
\int_\Omega \operatorname{rk} K(a,b)_{s\le t}\, g_p(s,t)\,\mathrm{d}s\mathrm{d}t &= \int_a^b \int_s^b p(p-1)(t-s)^{p-2}\,\mathrm{d}s\mathrm{d}t\\
&= \int_a^b p(b-s)^{p-1}\,\mathrm{d}s\\
&= (b-a)^p
\end{aligned}$$

$$= \|K(a,b)\|_p^p.$$

If a or b is not finite, then we still have the formal equality $\infty = \infty$.

If $X \cong \bigoplus_{i=1}^m K(a_i, b_i)$ is a persistence module with finitely many bars, then $\mathrm{rk}(X) = \sum_{i=1}^m \mathrm{rk}\, K(a_i, b_i)$ and the result follows from the case of a single bar by linearity of the Lebesgue integral. Since ephemeral bars do not contribute to the integral, it remains to prove the result for persistence modules $X \cong \bigoplus_{i=1}^\infty K(a_i, b_i)$ with countably many bars (see Remark C.8). In this case, the equality

$$\int_\Omega \sum_{i=1}^\infty \mathrm{rk}\, K(a_i, b_i)_{s\le t}\, g_p(s,t)\mathrm{d}s\mathrm{d}t = \sum_{i=1}^\infty \int_\Omega \mathrm{rk}\, K(a_i, b_i)_{s\le t}\, g_p(s,t)\mathrm{d}s\mathrm{d}t$$

follows from the Monotone Convergence Theorem [13, Thm. 2.14]. □

C.60 **Remark.** When $p = 1$, as observed for example in [7], the 1-norm $\|X\|_1$ of a persistence module coincides with the integral $\int_{\mathbf{R}} \dim X_t \mathrm{d}t$ of its Hilbert function $(\dim X)(t) := \dim X_t$. ■

C.61 **Theorem.** *Let $f \colon X \hookrightarrow Y$ be a monomorphism in* **Pfd** *and f_φ the bar-to-bar monomorphism induced by the canonical matching between X and Y. Then, for all $p \in [1,\infty]$,*

$$\|\operatorname{coker} f\|_p \ge \|\operatorname{coker} \Delta f\|_p \ge \|\operatorname{coker} f_\varphi\|_p.$$

Proof. For $p \in (1,\infty)$, we combine Propositions C.53 and C.59, along with the fact that the function $h \mapsto \int_\Omega h g_p$ is increasing.

For $p = 1$ and $p = \infty$, we conclude by continuity of p-norms with respect to p. □

C.62 **Remark.** When $p = 1$, the inequalities of Theorem C.61 are equalities:

$$\|\operatorname{coker} f\|_1 = \|\operatorname{coker} \Delta f\|_1 = \|\operatorname{coker} f_\varphi\|_1.$$

To see this, recall Remark C.60 and observe that for all $t \in \mathbf{R}$,

$$\dim(\operatorname{coker} f)_t = \dim(\operatorname{coker} \Delta f)_t = \dim(\operatorname{coker} f_\varphi)_t = \dim Y_t - \dim X_t.$$

For all $p > 1$, it is easy to find a monomorphism $f \colon X \hookrightarrow Y$ such that the inequalities of Theorem C.61 are strict. ■

Dually, we obtain the following result.

C.63 **Theorem.** *Let $f\colon X \twoheadrightarrow Y$ be an epimorphism in* **Pfd** *and f_φ the bar-to-bar epimorphism induced by the canonical matching between X and Y. Then, for all $p \in [1,\infty]$,*

$$\|\ker f\|_p \geq \|\ker(\Delta f^*)^*\|_p \geq \|\ker f_\varphi\|_p.$$

C.4.2 Recovering isometry result for Wasserstein distances

We now provide alternate proofs of the construction of algebraic p-Wasserstein distances and of the isometry result for p-Wasserstein distances [22]. We define algebraic p-Wasserstein distances by defining a noise system [21]: see [1] for the details to this approach in the context of *tame* persistence modules. In particular, we need to show the following result.

C.64 **Proposition.** *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in* **Pfd**. *Then, for all $p \in [1,\infty]$,*

1. *$\|X\|_p \leq \|Y\|_p$ and $\|Z\|_p \leq \|Y\|_p$ ("easy" axiom);*
2. *$\|Y\|_p \leq \|X\|_p + \|Z\|_p$ ("hard" axiom).*

To prove the "easy" axiom of noise systems we can proceed as follows, showing a stronger statement.

C.65 **Proposition.** *For every short exact sequence $0 \to X \to Y \to Z \to 0$ it holds that* $\mathrm{rk}(Y) \geq \mathrm{rk}(X) + \mathrm{rk}(Z)$.

Proof. Let $s \leq t$ in $\mathbf{R}$ and consider the appropriate diagram with two exact rows as in the snake lemma. We have

$$\mathrm{rk}\, X_{s\leq t} - \mathrm{rk}\, Y_{s\leq t} + \mathrm{rk}\, X_{s\leq t} = -\dim\ker X_{s\leq t} + \dim\ker Y_{s\leq t} - \dim\ker Z_{s\leq t} \leq 0,$$

where the equality follows from $\mathrm{rk}\, X_{s\leq t} = \dim X_t - \dim\ker X_{s\leq t}$ and similar identities for Y and Z, together with the identity $\dim X_t - \dim Y_t + \dim Z_t = 0$ by exactness of $0 \to X_t \to Y_t \to Z_t \to 0$; the inequality above follows from the exact sequence $0 \to \ker X_{s\leq t} \to \ker Y_{s\leq t} \xrightarrow{g} \ker Z_{s\leq t} \to \operatorname{coker} g \to 0$ (easy part of the snake lemma) again by Euler characteristic. □

By Proposition C.59 and monotonicity of the integral, we have the following consequence of Proposition C.65.

C.66 **Corollary** ([22, Remark 7.32]). *If* $p \in [1, \infty)$ *and* $0 \to X \to Y \to Z \to 0$ *is a short exact sequence, then*

$$\left(\|X\|_p^p + \|Z\|_p^p \right)^{\frac{1}{p}} \leq \|Y\|_p.$$

The "easy" axiom of noise systems, which states that $\|X\|_p \leq \|Y\|_p$ and $\|Z\|_p \leq \|Y\|_p$ follows immediately. The case $p = \infty$ follows by taking the limit.

We now prove the "hard" axiom of noise systems. This proof is the same as [1, Lemma 4.15]; the novelty is proving the result for **Pfd**.

C.67 **Proposition.** *For every short exact sequence* $0 \to X \to Y \to Z \to 0$ *it holds that* $\|Y\|_p \leq \|X\|_p + \|Z\|_p$.

Proof. Denote by $f\colon X \to Y$ the monomorphism in the short exact sequence. Then we have $Z \cong \operatorname{coker} f$.

First, let us consider the case where f is bar-to-bar. By definition (see Definition C.21), there exist barcode decompositions $X \cong \bigoplus_{i \in I} K(a_i, b_i)$ and $Y \cong \bigoplus_{j \in J} K_(c_j, d_j)$ and a matching $\mu \in \mathbf{Match}(I, J)$ such that

$$\operatorname{coker} f \cong \bigoplus_{(i,j) \in \mu} \operatorname{coker}(K(a_i, b_i) \to K(c_j, d_j)) \oplus \bigoplus_{j \in J \smallsetminus \pi_J(\mu)} K(c_j, d_j),$$

where, for (i,j) in μ, $b_i = d_j$. The p-norm of this persistence module is unaffected by ephemeral summands, so without loss of generality we assume that there are no ephemeral summands (for countability reasons). The p-norm of $\operatorname{coker} f$ is then equal to the p-norm of the countably long sequence $(\ell_j)_{j \in J}$ where, for all $j \in J$,

$$\ell_j = \begin{cases} \underbrace{(b_i - d_j)}_{=0} + c_j - a_i & \text{if there exists } (i,j) \text{ in } \mu, \\ d_j - c_j & \text{otherwise.} \end{cases}$$

Since f is a monomorphism, every $i \in I$ appears in this sequence. We also have $\|X\|_p = \|(b_i - a_i)_{i \in I}\|_p$ and $\|Y\|_p = \|(d_j - c_j)_{j \in J}\|_p$, and thus we obtain the desired inequality as the triangular inequality for the p-norm on infinite sequences of numbers.

Now let f be an arbitrary monomorphism, and let f_φ be the bar-to-bar monomorphism induced by the canonical matching between X and Y. By Theorem C.61, we conclude with

$$\|Y\|_p \leq \|X\|_p + \|\operatorname{coker} f_\varphi\|_p \leq \|X\|_p + \|\operatorname{coker} f\|_p = \|X\|_p + \|Z\|_p. \qquad \square$$

We provide an alternate (for one direction at least) proof to [22, Theorems 7.27-7.28], the isometry result between algebraic and combinatorial p-Wasserstein distances.

C.68 **Theorem.** *For any X and Y in* **Pfd**, *we have* $d_p^p(X,Y) = W_p(\mathsf{Dgm}(X), \mathsf{Dgm}(Y))$.

Proof. $\boxed{\leq}$ Let $\varphi\colon \mathsf{Dgm}(X) \to \mathsf{Dgm}(Y)$ be a matching. We want to construct a span with a smaller or equal cost. Following [22, Theorem 7.28], define

$$Z := \bigoplus_{((a,b),(c,d))\in\varphi} K(\max(a,c), \max(b,d))$$

and construct the natural span $X \leftarrow Z \to Y$. We can check that the cost of this span is exactly the cost of φ.

$\boxed{\geq}$ Let $X \xleftarrow{f} Z \xrightarrow{g} Y$ be a span. We want to construct a matching with a smaller or equal cost. First, factor the span as $X \hookleftarrow \operatorname{im} f \twoheadleftarrow Z \twoheadrightarrow \operatorname{im} g \hookrightarrow Y$ and replace each morphism with the morphism induced by canonical matchings. By Theorems C.61 and C.63, the cost of the new span $X \xleftarrow{f_b} Z \xrightarrow{g_b} Y$ is smaller than the cost of the original span.

The (strictly bar-to-bar) morphisms f_b and g_b induce matchings $\varphi_X\colon \mathsf{Dgm}(X) \to \mathsf{Dgm}(Z)$ and $\varphi_Y\colon \mathsf{Dgm}(Y) \to \mathsf{Dgm}(Z)$, respectively. Define

$$Z' := \bigoplus_{\substack{(a,b)\in\mathsf{Dgm}(X)\\ (c,d)\in\mathsf{Dgm}(Y)\\ \varphi_X(a,b)=\varphi_Y(c,d)}} K(\max(a,c), \max(b,d)).$$

Observe that f_b and g_b factor through Z'. This defines a new span (of strictly bar-to-bar morphisms) $X \leftarrow Z' \to Y$ with a smaller cost than the previous span: essentially, for each bar $\langle e,f\rangle$ in Z, we consider its matched bars $\langle a,b\rangle$ and $\langle c,d\rangle$ in X and Y, respectively, and we move the end-points of $\langle e,f\rangle$ as far left as possible while keeping f_b and g_b well-defined. This makes the cokernels and kernels of these two morphisms smaller.

By the discussion for the $\boxed{\leq}$ case, the cost of this span is equal to the cost of the matching $\psi = \varphi_X \circ \varphi_Y^{-1}$. Thus $\psi\colon \mathsf{Dgm}(X) \to \mathsf{Dgm}(Y)$ is a matching whose cost is smaller than the original span. This concludes the proof. □

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Paper D

Morse theory of Euclidean distance functions from algebraic hypersurfaces

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Morse theory of Euclidean distance functions from algebraic hypersurfaces

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Abstract

Let $Y \subseteq \mathbf{R}^n$ be a closed definable subset and $X \subseteq \mathbf{R}^n$ be a smooth manifold. We construct a version of Morse theory for the restriction to X of the Euclidean distance function from Y. This is done using the notion of critical points of Lipschitz functions and applying the theory of continuous selections. In this theory, nondegenerate critical points have two indices: a quadratic index (as in classical Morse theory), and a piecewise linear index (that relates to the notion of bottlenecks). This framework is flexible enough to simultaneously treat and unify the study of two cases of interest for computational algebraic geometry: bottlenecks and nearest point problems. We provide a technical toolset guaranteeing the applicability of the theory to the case where X, Y are generic algebraic hypersurfaces and use it to bound the number of critical points of the distance from Y restricted to X, among other applications.

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D.1 Introduction

D.1.1 A Morse theory for distance functions

Distance functions have been the object of extensive study over the years, classically in differential geometry [42, 52, 34, 54, 16, 13, 48, 20], and more recently in computational algebraic geometry [45, 23, 21, 12] and computational topology [15, 8, 46, 51, 3]. Various interesting properties of geometric objects in $\mathbf{R}^n$ (e.g. the Euclidean Distance Degree, bottlenecks) are defined in terms of Euclidean distance and arise as some type of "critical points", meaning that they are defined as set of solutions of algebraic systems of equations resembling the equations for critical points of a smooth function. Our goal here is to put these ideas into a general systematic framework, explaining how to recover all these notions at once using the language of Lipschitz geometry, and giving a topological interpretation of them.

Studying critical points of functions has many applications ranging from data analysis tasks such as the characterization of the texture of porous materials in [51] to machine learning where for example critical points of objective functions over Linear Convolutional Neural Networks are studied to understand their optimization [37, 38]. Low rank approximation problems for tensors as the one in Example D.1 are also relevant in machine learning as well as in signal processing and data compression [39, 17]. Furthermore bottlenecks can be used to produce provably dense samples of algebraic varieties, capturing the homology of data spaces naturally encoded by algebraic equations [22].

Given two real algebraic subsets $X, Y \subseteq \mathbf{R}^n$, in this paper we will explain how to construct a version of Morse theory for the function

$$\mathrm{dist}_Y\big|_X \colon X \to \mathbf{R},$$

where $\mathrm{dist}_Y(z) := \min_{y\in Y} \|y - z\|$ denotes the distance function from Y and $\mathrm{dist}_Y\big|_X$ denotes its restriction to X. For the applications we have in mind, both X and Y will be smooth algebraic hypersurfaces, but some of the results we will formulate are valid under weaker assumptions (e.g. in the definable[1] world or in the smooth world under some transversality conditions).

[1]In this paper, "definable" means definable in some o-minimal structure, for instance in the structure of globally subanalytic sets.

Compared to the classical theory, the key point here is that, even under the assumption that X and Y are smooth and algebraic, in general the function dist_Y is not differentiable, and similarly neither is $\mathrm{dist}_Y|_X$. What is then the definition of a critical point for functions of this type? If X and Y are semialgebraic, it is true that $\mathrm{dist}_Y|_X$ is semialgebraic, and one can at least set up a notion of "critical value" for it (see Section D.2.5), but this seems unsatisfactory: in particular, little of the geometry of distance functions is used. Instead, we base our approach on the notion of critical points of Lipschitz functions introduced by Clarke [19] and apply the more general Morse theory of continuous selections, as discussed in [1]. The applicability of this theory to the context of our interest requires some nondegeneracy conditions to be verified (as is the case for classical Morse theory!) and the technical part of the paper proves that these conditions are generically verified in the case of algebraic hypersurfaces (see Section D.4).

Our framework is flexible enough to include the two cases of interest for computational algebraic geometry: when X is $\mathbf{R}^n$ and Y is a smooth algebraic set, the critical points of $\mathrm{dist}_Y|_X$ (as defined below) leads to the notion of bottleneck Degree [23]; when Y reduces to a point and X is an algebraic set, they lead to the notion of Euclidean Distance Degree [24], see Section D.1.5 below.

In our exposition we focus on the case when either X or Y (or both) are smooth real algebraic **hypersurfaces**, and genericity is considered with respect to the defining polynomials. We also treat the case when X is $\mathbf{R}^n$ and Y is a finite set of points, which is relevant for applications.

D.1 **Example.** Given a point cloud in the space of tensors, what is the closest rank-one tensor to this point cloud? This problem can be formulated as finding the critical point with smallest value of $\mathrm{dist}_Y|_X$, where $Y = \{y_0, \ldots, y_m\}$ is the point cloud in the space of tensors $\mathbf{R}^{n_1} \otimes \cdots \otimes \mathbf{R}^{n_j}$ and X is the set of rank-one tensors. ■

We expect that most of the results extend to the case of complete intersections, but we leave this to future work. Instead, we provide a technical toolset (most notably a multijet parametric transversality theorem for polynomials, see Section D.4.1) which the reader will easily be able to apply in their case of interest.

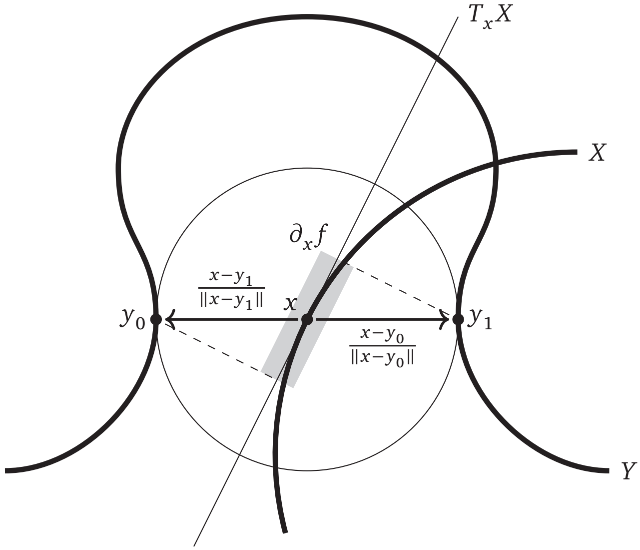

Figure D.1: Example of the subdifferential of $f = \mathrm{dist}_Y|_X$ at a critical point x. The subdifferential $\partial_x f$ is the interval between the projections of the vectors $\frac{x-y_0}{\|x-y_0\|}$ and $\frac{x-y_1}{\|x-y_1\|}$ onto T_xX.

D.1.2 Critical points of distance functions

Let us first explain how to define "critical points" in our context. The key observation is that the function $\mathrm{dist}_Y|_X$ is Lipschitz. Following Clarke [19], for such functions, one can define the notion of **subdifferential**. This is done for a general locally Lipschitz function $f\colon X \to \mathbf{R}$ as follows (see Definition D.11).

Denoting by $\Omega(f) \subseteq X$ the set of points where f is differentiable (of full measure by Rademacher's Theorem), the **subdifferential of** f **at** $x \in X$, denoted by $\partial_x f$, is the convex body

$$\partial_x f := \mathrm{co}\left\{ \lim_{x_k \to x, x_k \in \Omega(f)} D_{x_k} f \,\middle|\, \text{the limit exists} \right\} \subseteq (T_xX)^*.$$

It is often practical to assume that $X \subseteq \mathbf{R}^n$ is a Riemannian manifold with the metric induced by the Euclidean metric of $\mathbf{R}^n$, so that one can identify $T_xX \cong (T_xX)^*$ and view the subdifferential as a convex body in T_xX.

D.2 **Definition** (Critical points, Definition D.12). A **critical point** of f is a point $x \in X$ such that the subdifferential $\partial_x f$ contains the zero vector. ■

The set of critical points of f is denoted by $C(f)$. A regular point is a point that is not critical. The definitions of critical value and regular value are the usual ones. Minima and maxima are critical points; moreover, at a point of differentiability, one recovers the classical definitions. Going back to our functions of interest, if $Y \subseteq \mathbf{R}^n$ is a closed set and $x \in \mathbf{R}^n \setminus Y$, then the subdifferential of dist_Y at x can be easily described, see Corollary D.20:

$$\partial_x(\operatorname{dist}_Y) = \operatorname{co}\left\{ \frac{x-y}{\|x-y\|} \,\middle|\, y \in B(x, \operatorname{dist}_Y(x)) \cap Y \right\}, \tag{D.1}$$

where $B(x, \operatorname{dist}_Y(x))$ is the closed ball of radius $\operatorname{dist}_Y(x)$ centered in x. If X and Y are in general position (see Proposition D.21) The subdifferential of $\operatorname{dist}_Y|_X$ at $x \in X$ is just the projection of $\partial_x(\operatorname{dist}_Y)$ on the tangent space T_xX, see Figure D.1.

Various statements from classical analysis extend to the Lipschitz framework of distance functions. For instance, under the only assumption that Y is definable, the set of critical values of $\operatorname{dist}_Y|_X$ is definable[2] and of measure zero (this is called the "Sard property", see Section D.2.3). Another example of a classical result that extends to this framework is the Deformation Lemma, whose proof is based on Clarke's Implicit Function Theorem [19]. In order to state the lemma, we define for all $t \in \mathbf{R}$ the sublevel set

$$X_Y^t := \left\{ x \in X \,\middle|\, \operatorname{dist}_Y(x) \le t \right\}.$$

D.3 **Proposition** (Deformation Lemma, Lemma D.27). *If the interval $[a, b] \subset \mathbf{R}$ contains no critical values of* $\operatorname{dist}_Y|_X$*, then the set X_Y^b deformation retracts to the set X_Y^a.*

D.1.3 Nondegenerate critical points

In the definable framework one can even prove that the cohomology of X_Y^t changes in a controlled manner when passing a critical value (Proposition D.28) but, as in the classical theory, in order to be able to say more, one needs to introduce the notion of **nondegenerate** critical points. This is done using the theory of **continuous selections** as follows.

[2] Definability is needed, see Remark D.26.

Following [1], we say that $f\colon X \to \mathbf{R}$ is a continuous selection of the finite list $(f_0, \dots, f_m)$ of $\mathscr{C}^2$ functions if f is continuous and if for every $x \in X$ there exists $i \in \{0, \dots, m\}$ such that $f(x) = f_i(x)$. In this case we write $f \in \mathrm{CS}(f_0, \dots, f_m)$. For $f \in \mathrm{CS}(f_0, \dots, f_m)$ and $i \in \{0, \dots, m\}$, we denote by $S_i \subseteq X$ the set

$$S_i := \left\{ x \in X \,\middle|\, f(x) = f_i(x) \right\}.$$

Given $x \in X$, the **effective index set** of $f \in \mathrm{CS}(f_0, \dots, f_m)$ at $x \in X$ is defined as

$$I(x) := \left\{ i \in \{0, \dots, m\} \,\middle|\, x \in \overline{\mathrm{int}(S_i)} \right\}.$$

Note that continuous selections are locally Lipschitz, so we can talk about their critical points using the language introduced above. We can think of these functions as "piecewise $\mathscr{C}^2$ functions".

Now comes the crucial definition.

D.4 **Definition** (Nondegenerate critical points, Definition D.31). For a continuous selection $f \in \mathrm{CS}(f_0, \dots, f_m)$, a critical point $x \in C(f)$ is said to be **nondegenerate** if the following conditions are satisfied:

(i) For every $i \in I(x)$ the set of differentials $\{D_x f_j \mid j \in I(x) \smallsetminus \{i\}\}$ is linearly independent.

(ii) Denote by $(\lambda_i)_{i \in I(x)}$ the unique list of numbers such that

$$\sum_{i \in I(x)} \lambda_i = 1 \quad \text{and} \quad \sum_{i \in I(x)} \lambda_i D_x f_i = 0$$

and by $\lambda f\colon X \to \mathbf{R}$ the function defined by $\lambda f(z) := \sum_{i \in I(x)} \lambda_i f_i(z)$. Then the second differential of λf is nondegenerate on

$$V(x) := \bigcap_{i \in I(x)} \ker(D_x f_i). \qquad \blacksquare$$

When $f \in \mathrm{CS}(f_0)$, f is $\mathscr{C}^2$ and the above definition reduces to the usual definition of nondegenerate critical point, in the sense of Morse theory.

Compared to what happens with classical Morse functions, we see that a nondegenerate critical point of a continuous selection in general comes with two indices:

- the number $k(x) := \#I(x) - 1$, which we call the **piecewise linear index**;
- the number $\iota(x)$ of negative eigenvalues of the restriction of the second differential of λf to $V(x)$, which we call the **quadratic index**.

Continuous selections have normal forms near their nondegenerate critical points, in a fashion similar to the Morse Lemma [44, Lemma 2.2], see Remark D.33. Therefore, as for the classical Morse theory, for a continuous selection f the cohomology of the sublevel sets $\{f \leq t\}$ changes in a controlled way when passing a critical value with only nondegenerate critical points (see [1] and Proposition D.34), and this change can be bounded by the two numbers k and ι only. For instance, in our context, we have the following result.

D.5 **Proposition** (Corollary D.41). *If* $\mathrm{dist}_Y|_X$ *is a continuous selection at* $x \in X$ *and* x *is a nondegenerate critical point for it such that* $C(\mathrm{dist}_Y|_X) \cap (\mathrm{dist}_Y|_X)^{-1}(c) = \{x\}$*, then for all* $\epsilon > 0$ *small enough,*

$$H^*(X_Y^{c+\varepsilon}, X_Y^{c-\varepsilon}) \cong \tilde{H}^*(S^{k(x)+\iota(x)}).$$

D.1.4 Critical points and bottlenecks

In the statement of Proposition D.5, the assumption that $\mathrm{dist}_Y|_X$ is a continuous selection is essential. Indeed, in general, distance functions might not fall in this class, even under seemingly nice assumptions, see Example D.37. We show, however, that this is generically the case in Corollary D.63.

Note that, due to the description (D.1), in order to satisfy the condition (i) from Definition D.4 we need in particular that, at each critical point $x \in C(\mathrm{dist}_Y|_X)$, the function $m_Y\colon \mathbf{R}^n \to \mathbf{R}$, defined by

$$m_Y(x) := \#\Big(B(x, \mathrm{dist}_Y(x)) \cap Y\Big),$$

should be bounded by $n+1$. In this direction, Yomdin proved in [54, Proposition 3] that this is true for a generic embedding $i\colon Y \to \mathbf{R}^n$, when Y is a compact smooth manifold. Here we prove the following result, which is an analogue of Yomdin's result in the algebraic framework (but does not follow from it). For $d \in \mathbf{N}$, we denote by $\mathscr{P}_d$ the set of polynomials on $\mathbf{R}^n$ of degree at most d.

D.6 **Theorem** (Theorem D.56). *For generic $q \in \mathscr{P}_d$ with $d \geq 2$, writing $Y = Z(q)$, and for every $x \in \mathbf{R}^n$, the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a nondegenerate simplex with at most $n+1$ points. In particular, $m_Y \leq n+1$.*

Now let $Y \subseteq \mathbf{R}^n$ be an algebraic set. Following [21], one says that a critical point $x \in C(\mathrm{dist}_Y)$ is a **geometric** $(k+1)$**-bottleneck**, also called bottleneck of order k, if x is a convex combination of $k+1$ affinely independent points in $B(x, \mathrm{dist}_Y(x)) \cap Y$ and is not a convex combination of a smaller number of points in $B(x, \mathrm{dist}_Y(x)) \cap Y$. In particular, a geometric $(k+1)$-bottleneck x satisfies $m_Y(x) \geq k+1$. Under the assumption that dist_Y is a continuous selection with only nondegenerate critical points (this is generically true, see Theorem D.69), geometric $(k+1)$-bottlenecks are precisely critical points of dist_Y with piecewise linear index equal to k (see Definition D.4) and an analogue of the cellular decomposition theorem is Corollary D.41.

Motivated by the connection between the order of geometric bottlenecks and the piecewise linear index, we introduce the following general definition, which, in the nondegenerate case, partitions the critical points according to their piecewise linear index.

D.7 **Definition** (k-critical points, Definition D.57). Let $X \subseteq \mathbf{R}^n$ be a smooth manifold and $Y \subseteq \mathbf{R}^n$ be a closed definable set. For all $k \geq 0$, we define the **k-critical points of** $\mathrm{dist}_Y|_X$ as the critical points $x \in X$ of $\mathrm{dist}_Y|_X$ such that $B(x, \mathrm{dist}_Y(x)) \cap Y = \{y_0, \ldots, y_k\}$. We denote by $C_{k,*}(\mathrm{dist}_Y|_X)$ the set of k-critical points[3] of $\mathrm{dist}_Y|_X$ not in Y.

Note that if $X \cap Y \neq \varnothing$, then every point in this intersection is critical for $\mathrm{dist}_Y|_X$, in fact these are precisely the global minima.

In the nondegenerate case (so that the quadratic and piecewise linear indices are defined), k-critical points have piecewise linear index k. We denote by $C_{k,\iota}(\mathrm{dist}_Y|_X) \subseteq C_{k,*}(\mathrm{dist}_Y|_X)$ the set of nondegenerate critical points of $\mathrm{dist}_Y|_X$ with piecewise linear index k and quadratic index ι. For instance, a geometric $(k+1)$-bottleneck that is nondegenerate with quadratic index ι is contained in $C_{k,\iota}(\mathrm{dist}_Y)$. ■

[3] At these critical points the quadratic index might not be defined, as they are not necessarily nondegenerate.

In the generic algebraic case, by Theorem D.6, every critical point is a k-critical point for some $k \in \{0, \dots, n\}$. In fact, a stronger result is true. The next theorem collects various statements from Section D.4 and shows that, in the generic case, all of the ideas that we have introduced so far combine to give a Morse theory for the function $\mathrm{dist}_Y|_X$ in the case X, Y are smooth algebraic hypersurfaces.

D.8 **Theorem.** *Let $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$ be generic polynomials, and let $X = Z(p)$ and $Y = Z(q)$ be their zero sets. Then, depending on the values of $d_1 = \deg(X)$ and $d_2 = \deg(Y)$ the following is true:*

- *if $d_1 \geq 3$ and $d_2 \geq 4$, then the function $\mathrm{dist}_Y|_X$ is a continuous selection near each of its critical points (Corollary D.63);*
- *if $d_1, d_2 \geq 4$, then every critical point of $\mathrm{dist}_Y|_X$ not in Y is nondegenerate (Theorem D.69); in particular Proposition D.5 can be applied;*
- *if $d_1, d_2 \geq 3$, then for every $k \in \{1, \dots, n\}$, the number of k-critical points of $\mathrm{dist}_Y|_X$ not in Y can be bounded by*

$$\#C_{k,*}(\mathrm{dist}_Y|_X) \leq c(k,n)\deg(X)^n \deg(Y)^{n(k+1)}, \tag{D.2}$$

for some $c(k,n) > 0$ depending on n and k only (Theorem D.60).

D.9 **Remark.** In the case of $(X, Y) = (\mathbf{R}^n, Y)$, which is not covered by the above theorem, the result still applies, see Remark D.70. The fact that, for every $k \in \{1, \dots, n\}$, there are generically finitely many k-critical points not in Y follows from [21, Theorem 4.14] (which holds for non-quadratic complete intersections). However, [21, Theorem 4.14] does not prove that this number is zero for $k \geq n+1$ (i.e. it doesn't rule out the possibility of critical points with m_Y arbitrarily large), which is instead a consequence of Theorem D.6. ■

D.10 **Remark** (A duality formula). The partition of nondegenerate critical points according to the *sum* of their piecewise linear and quadratic index provides an interesting duality formula (see Corollary D.44), that for the generic pair (X, Y), if X and Y are compact, reads as

$$\chi(Y) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \#C_{k,\iota}(X,Y) = \chi(X) + \sum_{k,\iota \geq 0} (-1)^{k+\iota} \#C_{k,\iota}(Y,X).$$

For example, when $Y = \{y\}$ with $y \notin X$, this formula gives back the Euler Characteristic of X as the alternating sum of the number of critical points of quadratic index ι. ■

D.1.5 Related works on Morse theory of distance functions

Prior to our work there have been several contributions to build a framework for the Morse theory of distance functions: most notably [16], that deals with distance-from-a-point functions on Riemannian manifolds, and [8], which treats the case of point clouds.

We note that, in [8], the piecewise linear index k is identified as an analogue of the classical Morse index in the case of distance functions from a finite set of points $Y = \{y_0, \dots, y_m\}$, and in [21, Remark 4.2] it is compared to the order of the geometric bottlenecks in the context of more general distance functions (with a shift in index). However, as we explained above, in the general case the piecewise linear index k is not the correct analogue of the classical Morse index. It is true that, for the distance from a finite set of points, the piecewise linear index correctly describes the changes in topology, but this is because the quadratic index in this case is always zero. In general, as demonstrated by Proposition D.5, changes in topology take into account both the piecewise linear index and the quadratic index.

More recently, the authors of [51] developed the Morse theory of signed distance functions from a hypersurface in Euclidean space. Via an argument based on transversality theory, they prove that, for a generic embedding of a smooth compact surface in $\mathbf{R}^3$, the associated signed distance function has finitely many critical points that are all nondegenerate. The Morse theory for distance functions we present in Section D.2 is similar to the treatment in [51, Sect. 3]. The main difference between the two approaches is that the authors of [51] build on the theory of Min-type functions [31] which, like continuous selections, admit normal forms at nondegenerate critical points. For applications to Euclidean distance functions, the generalized Morse theory built on the theory of Min-type functions is equivalent to that built on continuous selections, as we observe in Remark D.35.

D.1.6 The bottleneck Degree, the Euclidean Distance Degree and related notions

Let us go back to the study of geometric $(k+1)$-bottlenecks in details. As we have seen, they are in particular critical points of dist_Y with at least $k+1$ closest points in Y; in the nondegenerate case, this means critical points with piecewise linear index k. When X is $\mathbf{R}^n$ and Y is an algebraic hypersurface, the above result can still be applied (see Remark D.70 and Remark 1.4.4), and the bound (D.2) becomes

$$\#C_{k,*}(\mathrm{dist}_Y) \leq c(k,n)(\deg(Y))^{n(k+1)}.$$

When $k=1$, geometric 2-bottlenecks are simply called "bottlenecks". These bottlenecks are easier to study, because of the structure of their equations (roughly speaking, they do not involve quantifier elimination). The **bottleneck degree** of a smooth variety Y defined over $\mathbf{R}$, denoted by $\mathrm{BND}(Y)$, was defined in [23] as the number of complex solutions of this system of equations for generic Y. The motivation for the study in [23] was to control the number of bottlenecks of (the real part of) a real variety Y, and this can be done passing to the complexification and using the bound

$$\#C_{1,*}(\mathrm{dist}_Y) \leq \mathrm{BND}(Y).$$

In [23], the authors compute various bottleneck degrees (planar and space curves, space surfaces, and threefolds) and deduce a bound that, for these examples, is of the form $\#C_{1,*}(\mathrm{dist}_Y) \leq \tilde{c}(1,n)d^{2n}$. In this sense, our bound (D.2) generalizes this for all dimensions and for bottlenecks of all orders.

A similar discussion holds for the notion of **Euclidean distance degree** of a smooth real algebraic variety X, denoted by $\mathrm{EDD}(X)$. This was defined in [24] as the number of complex solutions of the equations describing the critical points of the distance function from a generic point $y \in \mathbf{R}^n$.

If one is only interested in the real solutions to the critical points equations, using our language, one could define the following notion. For generic pairs (X,Y) of real algebraic sets in $\mathbf{R}^n$, we define the **Real Euclidean Distance Degree of** X **relative to** Y as the number

$$\mathrm{EDD}_{\mathbf{R}}(X,Y) := \#C(\mathrm{dist}_Y|_X).$$

In the special cases $(X,Y) = (\mathbf{R}^n, Y)$ and $(X,Y) = (X,\{y\})$, one could refine this definition and obtain the quantities

$$\mathrm{BND}_{\mathbf{R}}(Y) := \mathrm{EDD}_{\mathbf{R}}(\mathbf{R}^n, Y) \quad \text{and} \quad \mathrm{EDD}_{\mathbf{R}}(X) := \mathrm{EDD}_{\mathbf{R}}(X,\{y\}).$$

In the generic case, as we have seen, both these numbers are actually the number of critical points of an appropriate function (the function $\mathrm{dist}_Y|_X$) and can be bounded by (D.2).

Note that, unlike analogue complex notions, this number depends on the pair (X,Y). It is clear from this discussion, for instance, that $\mathrm{EDD}_{\mathbf{R}}(X) \le \mathrm{EDD}(X)$. However, we leave the study of this notion of "relative EDD" from the complex point of view to future investigations. The definition of the complex solutions of the system defining our critical points requires some care, because convexity is used, but the constructions from Section D.4 also work over the complex numbers.

A natural question for the experts is: after it is defined, how to compute the (complex) Euclidean Distance Degree of X relative to Y using polar classes?

D.1.7 Tools for genericity

Our proof uses *real techniques* as opposed to, for example, the techniques from [23], which come from complex algebraic geometry and use the notion of polar classes. However, the technical results from the current paper (in particular, those from Section D.4.1) extend easily to the complex world, with the same proofs. In fact, we propose these techniques as easy-to-use substitute tools to verify that codimensions are the expected ones in the generic case. For example, the proof of finiteness of the number of $(k+1)$-bottlenecks for non-quadratic complete intersections from [21, Theorem 4.14] uses the Alexander-Hirschowitz Theorem, that here is substituted with the parametric multijet theorem (Theorem D.54).

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D.2 Morse theory for distance functions

Let $Y \subseteq \mathbf{R}^n$ be a closed definable subset and $X \subseteq \mathbf{R}^n$ a smooth manifold. In this section we will explain how to construct a version of Morse theory for the function

$$\mathrm{dist}_Y\big|_X \colon X \to \mathbf{R},$$

where $\mathrm{dist}_Y(z) := \min_{y\in Y} \|y - z\|$ and $\mathrm{dist}_Y\big|_X$ denotes the restriction of dist_Y to X. Except for some special cases (e.g. when $Y = \{y\}$ is a point which is not in X), the function $\mathrm{dist}_Y\big|_X$ is not differentiable and not convex. To formulate basic statements about the change of the topology of its sublevel sets, one needs to introduce first appropriate notions of critical points and critical values. This will be done, first for the function dist_Y and then for $\mathrm{dist}_Y|_X$, following a general theory for locally Lipschitz functions.

D.2.1 Critical points

An appropriate notion of critical points for dist_Y is introduced as follows. First we observe that dist_Y is Lipschitz, and therefore differentiable almost everywhere by Rademacher's Theorem. Following Clarke [19], one can therefore define the **subdifferential** of dist_Y at a point $x \in \mathbf{R}^n$.

D.11 **Definition** (Subdifferential). Let (X, g) be a Riemannian manifold and $f\colon X \to \mathbf{R}$ be a locally Lipschitz function. (In the framework we have in mind $X \subseteq \mathbf{R}^n$ is a submanifold with the induced Riemannian metric from the ambient Euclidean space.) We denote by $\Omega(f) \subseteq X$ the set of differentiability points of f (of full measure by Rademacher's Theorem). The **subdifferential of f at** $x \in X$, denoted by $\partial_x f$, is the convex body defined by

$$\partial_x f := \mathrm{co}\left\{ \lim_{x_k \to x, x_k \in \Omega(f)} \nabla f(x_k) \,\middle|\, \text{the limit exists} \right\} \subseteq T_x X.$$

Note that at a strictly differentiable point, the subdifferential reduces to a singleton $\partial_x f = \{\nabla f(x)\}$. ■

D.12 **Definition** (Critical points). We say that $x \in X$ is a **critical point** of a locally Lipschitz function $f\colon X \to \mathbf{R}$ if 0 belongs to $\partial_x f$. We denote by $C(f) \subseteq X$ the set of critical points of f.

A point $x \in X$ is said to be a **regular point** of f if it is not critical; a **critical value** of f is a real number $c \in \mathbf{R}$ such that $f^{-1}(c) \cap C(f) \neq \varnothing$ and a **regular value** of f is a real number $c \in \mathbf{R}$ which is not critical. ■

D.13 **Remark.** For example, local maxima and local minima are critical points of locally Lipschitz functions [49]. ■

D.14 **Definition** (Medial axis). Let $Y \subseteq \mathbf{R}^n$ be a closed set. We introduce the function $m_Y\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$, defined for $z \in \mathbf{R}^n$ by

$$m_Y(z) := \#\left(B(z, \mathrm{dist}_Y(z)) \cap Y\right),$$

where $B(z, \mathrm{dist}_Y(z))$ denotes the closed ball of radius $\mathrm{dist}_Y(x)$ centered in x, and the set $M_Y \subset \mathbf{R}^n$, called the **medial axis** of Y, defined by

$$M_Y := \left\{ z \in \mathbf{R}^n \,\middle|\, m_Y(z) \geq 2 \right\}. \qquad \blacksquare$$

D.15 **Remark.** [54, Prop. 3] proves that m_Y is finite if $Y = i(W)$ is the image of a compact smooth manifold W by a generic embedding $i \colon M \to \mathbf{R}^n$. An analogue result in the algebraic setting is shown by Theorem D.56. ■

D.16 **Lemma.** *Let $Y \subseteq \mathbf{R}^n$ be a closed definable set. The set M_Y is also definable, of codimension at least one, and the function* dist_Y *is continuously differentiable on $\mathbf{R}^n \setminus (Y \cup \overline{M_Y})$.*

Proof. The fact that M_Y is definable is obvious. The fact that it has codimension at least one follows from the strict convexity of the norm and the fact that dist_Y is continuously differentiable on $\mathbf{R}^n \setminus (Y \cup \overline{M_Y})$) is the content of [28, Theorem 4.8(5)]. □

D.17 **Example** (Critical points of the distance function from a finite set). Let $Y = \{y_0, \dots, y_m\}$ be a finite set of points in $\mathbf{R}^n$, and let $X = \mathbf{R}^n$.

The work of Bobrowski and Adler [8] explains the Morse theory for the function dist_Y; it turns out that this is a special case of the Morse theory from this paper.

To each $y_i \in Y$ one can associate the **Voronoi cell** $V(y_i) := \{x \in \mathbf{R}^n \mid \|x - y_i\| \leq \|x - y_j\|, \forall y_j \in Y\}$. Voronoi cells are convex polyhedra, and any two different Voronoi cells have disjoint interiors. The collection of Voronoi cells and their faces constitute a polyhedral complex [55] called the **Voronoi diagram** of Y, see e.g. [9], which covers the whole space $\mathbf{R}^n$.

In the interior of a Voronoi cell $V(y_i)$, the distance function dist_Y coincides with $\mathrm{dist}_{\{y_i\}}$ and its only critical point is y_i. The medial axis M_Y is union of the boundaries of all Voronoi cells, and it contains all remaining critical points of dist_Y (cf. Lemma D.16). If no subset of $n+2$ points of Y lie on a common hypersphere, then for each $x \in \mathbf{R}^n$ the value $m_Y(x)$ coincides with $n+1$ minus the lowest dimension of a Voronoi face containing x.

A set $Y = \{y_0, \dots, y_m\}$ in $\mathbf{R}^n$ is in **general position** if every subset of up to $n+1$ points of Y is affinely independent. If Y is in general position, then one can find the critical points of dist_Y via the following procedure: for each subset Y' of $k+1$ points in Y, with $k \leq n$, let x denote the center of the unique $(k-1)$-sphere containing Y' and check whether

(i) $x \in \mathrm{co}(Y')$, and
(ii) there is no point $y \in Y \setminus Y'$ such that $\|x-y\| < \mathrm{dist}_{Y'}(x)$.

The point x is critical if, and only if, the conditions (i) and (ii) are verified. Under the general position assumption on Y, the procedure we described gives an upper bound of $\sum_{k=0}^{n} \binom{m}{k+1}$ on the number of critical points of dist_Y.

(We continue the discussion of this example below, after we have introduced the notion of nondegenerate critical points, see Example D.47.) ■

D.2.2 Subdifferentials of distance functions via Danskin's theorem

We now describe the Clarke subdifferential of the distance function in a more direct way. The key point is that the squared distance from a closed set can be written as the minimum of a family of smooth functions. This allows one to apply Danskin's theorem and obtain an explicit formula for the subdifferential, both in the ambient space and after restriction to a smooth submanifold.

We first recall the version of Danskin's theorem that we need.

D.18 **Proposition** (Danskin's theorem). *Let $W \subseteq \mathbf{R}^m$ be compact, and let $\phi\colon \mathbf{R}^n \times W \to \mathbf{R}$ be continuous. Assume that for every $w \in W$ the function $\phi(-,w)\colon \mathbf{R}^n \to \mathbf{R}$ is differentiable, and that for every $x \in \mathbf{R}^n$ the map*

$$w \mapsto \nabla_x \phi(x,w)$$

is continuous on W. Define

$$f(x) := \max_{w \in W} \phi(x,w).$$

Then

$$\partial_x f = \mathrm{co}\left\{\nabla_x \phi(x,z) \mid z \in Z(x)\right\},$$

where

$$W(x) := \left\{ w \in W \,\middle|\, \phi(x,w) = \max_{w \in W} \phi(x,w) \right\}.$$

Proof. This is [7, Prop. B.22(b)]. □

We begin with the ambient case.

D.19 **Proposition.** *Let $Y \subseteq \mathbf{R}^n$ be closed, and define*

$$\eta_Y(x) := \frac{1}{2}\operatorname{dist}_Y(x)^2.$$

Then, for every $x \in \mathbf{R}^n$,

$$\partial_x(\eta_Y) = \operatorname{co}\{x - y \mid y \in B(x, \operatorname{dist}_Y(x)) \cap Y\}.$$

Proof. Fix $x_0 \in \mathbf{R}^n$. Since Y is closed, for every $x \in \mathbf{R}^n$ the set $B(x, \operatorname{dist}_Y(x)) \cap Y$ is nonempty and compact.

We claim that there exist a neighborhood U of x_0 and $R > 0$ such that, for every $x \in U$,

$$B(x, \operatorname{dist}_Y(x)) \cap Y \subseteq B(0, R).$$

Indeed, choose U bounded. Since dist_Y is 1-Lipschitz, it is bounded on U, and therefore

$$R := \sup_{x \in U} \|x\| + \sup_{x \in U} \operatorname{dist}_Y(x)$$

is finite. If $x \in U$ and $y \in B(x, \operatorname{dist}_Y(x)) \cap Y$, then

$$\|y\| \leq \|y - x\| + \|x\| = \operatorname{dist}_Y(x) + \|x\| \leq R.$$

This proves the claim.

Set

$$W := Y \cap B(0, R), \qquad \phi(x, y) := \langle x, y\rangle - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2}.$$

For $x \in U$, one has

$$\eta_Y(x) = \min_{y \in Z} \frac{1}{2}\|x - y\|^2,$$

or equivalently

$$-\eta_Y(x) = \max_{y \in Z} \phi(x, y).$$

Moreover, W is compact, ϕ is continuous, for every $y \in W$ the function $\phi(-, y)$ is smooth, and

$$\nabla_x \phi(x, y) = y - x$$

depends continuously on y. Hence Proposition D.18 applies and yields

$$\partial_x(-\eta_Y) = \operatorname{co}\{y - x \mid y \in W(x)\},$$

where, as above,

$$W(x) = \left\{ y \in W \,\middle|\, \phi(x,y) = \max_{w \in W} \phi(x,w) \right\}.$$

By construction, the maximizing points are exactly the nearest points to x in Y, that is,

$$W(x) = B(x, \mathrm{dist}_Y(x)) \cap Y.$$

Therefore

$$\partial_x(-\eta_Y) = \mathrm{co}\left\{ y - x \mid y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\},$$

and multiplying by -1 gives

$$\partial_x(\eta_Y) = \mathrm{co}\left\{ x - y \mid y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}. \qquad \square$$

D.20 **Corollary.** *Let $Y \subseteq \mathbf{R}^n$ be closed and let $x \in \mathbf{R}^n \smallsetminus Y$. Then*

$$\partial_x(\mathrm{dist}_Y) = \mathrm{co}\left\{ \frac{x-y}{\|x-y\|} \,\middle|\, y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}.$$

Proof. Since $x \notin Y$, one has $\mathrm{dist}_Y(x) > 0$. Moreover,

$$\eta_Y = \frac{1}{2} \mathrm{dist}_Y^2,$$

hence at every differentiability point of dist_Y one has

$$\nabla \eta_Y = \mathrm{dist}_Y \nabla \mathrm{dist}_Y.$$

Passing to the Clarke subdifferential at x, this gives

$$\partial_x(\eta_Y) = \mathrm{dist}_Y(x)\, \partial_x(\mathrm{dist}_Y).$$

Using Proposition D.19, we obtain

$$\partial_x(\mathrm{dist}_Y) = \frac{1}{\mathrm{dist}_Y(x)} \mathrm{co}\left\{ x - y \mid y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}.$$

Since $\|x-y\| = \mathrm{dist}_Y(x)$ for every $y \in B(x, \mathrm{dist}_Y(x)) \cap Y$, this becomes

$$\partial_x(\mathrm{dist}_Y) = \mathrm{co}\left\{ \frac{x-y}{\|x-y\|} \,\middle|\, y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}. \qquad \square$$

We now turn to the restriction of the distance function to a smooth submanifold.

D.21 **Proposition.** *Let $Y \subseteq \mathbf{R}^n$ be closed and let $X \subseteq \mathbf{R}^n$ be a smooth manifold. Then, for every $x \in X \setminus Y$,*

$$\partial_x(\mathrm{dist}_Y|_X) = \mathrm{proj}_{T_xX}\big(\partial_x(\mathrm{dist}_Y)\big),$$

where $\mathrm{proj}_{T_xX}\colon \mathbf{R}^n \to T_xX$ *denotes the orthogonal projection. More explicitly,*

$$\partial_x(\mathrm{dist}_Y|_X) = \mathrm{co}\left\{ \frac{\mathrm{proj}_{T_xX}(x-y)}{\|x-y\|} \,\middle|\, y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\}.$$

Proof. Fix $x_0 \in X \setminus Y$, and choose a smooth chart

$$\psi\colon U \subseteq \mathbf{R}^{\dim X} \to X$$

such that $\psi(0) = x_0$.

As in the proof of Proposition D.19, after shrinking U if necessary there exists $R > 0$ such that

$$B(\psi(u), \mathrm{dist}_Y(\psi(u))) \cap Y \subseteq B(0,R) \qquad \text{for every } u \in U.$$

Set

$$W := Y \cap B(0,R)$$

and define

$$F(u) := -\frac{1}{2}\,\mathrm{dist}_Y|_X(\psi(u))^2.$$

Then

$$F(u) = \max_{y\in W} \Phi(u,y),$$

where

$$\Phi(u,y) := \langle \psi(u), y\rangle - \frac{\|\psi(u)\|^2}{2} - \frac{\|y\|^2}{2}.$$

The function Φ is continuous, for every $y \in Z$ the function $\Phi(-,y)$ is smooth, and

$$\nabla_u \Phi(u,y) = D\psi(u)^\top (y - \psi(u))$$

depends continuously on y. Therefore Proposition D.18 applies and gives

$$\partial_0 F = \mathrm{co}\left\{ D\psi(0)^\top (y - x_0) \mid y \in B(x_0, \mathrm{dist}_Y(x_0)) \cap Y \right\}. \tag{D.3}$$

Moreover, by the chain rule, we have

$$\partial_0 F = D\psi(0)^\top \partial_{x_0}\left(-\frac{1}{2}\mathrm{dist}_Y|_X^2\right). \tag{D.4}$$

Identifying $T_{x_0}X$ with the image of $D\psi(0)$, Equations (D.3) and (D.4) give the following equality between projections onto $T_{x_0}X$:

$$\mathrm{proj}_{T_{x_0}}\partial_{x_0}\left(-\frac{1}{2}\mathrm{dist}_Y|_X^2\right) = \mathrm{co}\left\{\mathrm{proj}_{T_{x_0}X}(y-x_0)\;\middle|\; y \in B(x_0, \mathrm{dist}_Y(x_0)) \cap Y\right\}.$$

The subdifferential $\partial_{x_0}\left(-\frac{1}{2}\mathrm{dist}_Y|_X^2\right)$ is already in $T_{x_0}X$ by construction, so we can omit the projection on the lefthand side. Taking the opposite sign, we get

$$\partial_{x_0}\left(\frac{1}{2}\mathrm{dist}_Y|_X^2\right) = \mathrm{co}\left\{\mathrm{proj}_{T_{x_0}X}(x_0-y)\;\middle|\; y \in B(x_0, \mathrm{dist}_Y(x_0)) \cap Y\right\}.$$

Since $x_0 \notin Y$, one has $\mathrm{dist}_Y|_X(x_0) = \mathrm{dist}_Y(x_0) > 0$, and therefore

$$\partial_{x_0}(\mathrm{dist}_Y|_X) = \frac{1}{\mathrm{dist}_Y|_X(x_0)}\partial_{x_0}\left(\frac{1}{2}\mathrm{dist}_Y|_X^2\right).$$

Using that $\|x_0 - y\| = \mathrm{dist}_Y(x_0)$ for every $y \in B(x_0, \mathrm{dist}_Y(x_0)) \cap Y$, we conclude that

$$\partial_{x_0}(\mathrm{dist}_Y|_X) = \mathrm{co}\left\{\frac{\mathrm{proj}_{T_{x_0}X}(x_0-y)}{\|x_0-y\|}\;\middle|\; y \in B(x_0, \mathrm{dist}_Y(x_0)) \cap Y\right\}.$$

By Corollary D.20, this is exactly

$$\partial_{x_0}(\mathrm{dist}_Y|_X) = \mathrm{proj}_{T_{x_0}X}(\partial_{x_0}(\mathrm{dist}_Y)),$$

because orthogonal projection is linear. □

D.22 **Corollary.** *Let $Y \subseteq \mathbf{R}^n$ be closed and let $X \subseteq \mathbf{R}^n$ be a smooth manifold.*

1. *If $x \in X \cap Y$, then x is critical for* $\mathrm{dist}_Y|_X$.
2. *If $x \in X \setminus Y$, then x is critical for* $\mathrm{dist}_Y|_X$ *if and only if*

$$\partial_x(\mathrm{dist}_Y) \cap N_x X \neq \varnothing,$$

where $N_x X := (T_x X)^\perp$.

Proof. For the first statement, assume that $x \in X \cap Y$. If $\partial_x(\mathrm{dist}_Y|_X)$ did not include 0, then we could apply Clarke's inverse function theorem for locally Lipschitz functions [18, Theorem 1] and deduce that the function $\mathrm{dist}_Y|_X$ changes signs around x, which is impossible as it is always nonnegative. We deduce that $0 \in \partial_x(\mathrm{dist}_Y|_X)$, i.e., x is critical.

Now let $x \in X \setminus Y$. By Proposition D.21,

$$\partial_x(\mathrm{dist}_Y|_X) = \mathrm{proj}_{T_xX}(\partial_x(\mathrm{dist}_Y)).$$

Therefore x is critical for $\mathrm{dist}_Y|_X$ if and only if

$$0 \in \mathrm{proj}_{T_xX}(\partial_x(\mathrm{dist}_Y)).$$

Since $\partial_x(\mathrm{dist}_Y)$ is convex and proj_{T_xX} is linear, this is equivalent to the existence of some $v \in \partial_x(\mathrm{dist}_Y)$ such that

$$\mathrm{proj}_{T_xX}(v) = 0.$$

The latter equality is equivalent to $v \in N_xX$, and the second statement follows. □

D.23 **Remark.** In Propositions D.19 and D.21 and Corollaries D.20 and D.22, the only assumptions are that Y is closed and that X is smooth. In particular, definability and transversality are not used here. ■

D.24 **Example.** As an example, see Figure D.2, where X is a sphere and $Y = \{y_0, y_1\}$ consists of two points in $\mathbf{R}^3$. X intersects M_Y in a circle, and for every point x of this intersection, the subdifferential of dist_Y at x is the segment $[\frac{x-y_0}{\|x-y_0\|}, \frac{x-y_1}{\|x-y_1\|}]$. However, only when x is x_0 or x_1 does the normal space N_xX intersect the subdifferential: thus, of all the points of X on the medial axis, only x_0 and x_1 are critical. Of course, these are not the only critical points; the points minimizing $\mathrm{dist}_Y|_X$ are also critical points, and they do not lie on the medial axis. ■

D.2.3 The Sard property

If $X \subseteq \mathbf{R}^n$ is smooth, the fact that the set of critical values of $\mathrm{dist}_Y|_X$ has zero measure follows from a result of Rifford [48], which generalizes the Morse-Sard theorem for smooth functions to the case of distance functions. In the definable context we can actually use the following result.

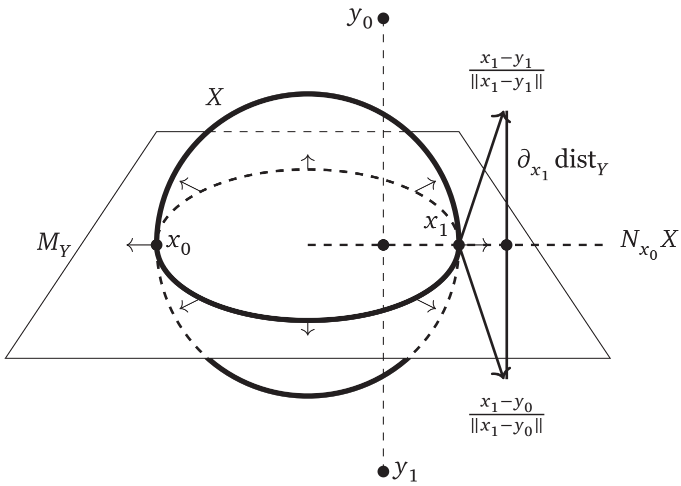


Figure D.2: Example in $\mathbf{R}^3$ of the characterization of critical points using normal spaces. X is a sphere and $Y = \{y_0, y_1\}$ consists of two points. The plane represents the medial axis M_Y; the small arrows represent normal vectors from various points of X; $N_{x_1}X$ (the horizontal dashed line) is the normal space of X at x_1.

D.25 **Proposition** ([10, Theorem 7]). *Let $f\colon X \to \mathbf{R}$ be a locally Lipschitz and definable function. Then f is constant on each component of the set of its critical points and therefore the set of critical values of f is finite. In particular, if $X, Y \subseteq \mathbf{R}^n$ are definable sets with X smooth, the set of critical values of* $\mathrm{dist}_Y|_X$ *is finite.*

D.26 **Remark.** The requirement that Y is a definable set seems necessary to avoid pathological examples. For instance, as stated in [48, Remark 1.1], while the set of all critical values of the distance function from a compact subset in $\mathbf{R}^3$ is of measure zero [30], counterexamples exist in higher dimensions [29, Example 3] (see also [53]). ■

D.2.4 The deformation lemma

Given $f\colon X \to \mathbf{R}$ and $t \in \mathbf{R}$, we denote by $X^t := \{x \in X \mid f(x) \leq t\}$ and in the special case $f = \mathrm{dist}_Y|_X$ we set

$$X_Y^t := \left\{ x \in X \,\middle|\, \mathrm{dist}_Y(x) \leq t \right\}.$$

We are interested in understanding how the topology changes letting $t \in \mathbf{R}$ vary. This can be measured by the relative cohomology groups $H^*(X_Y^{t+\varepsilon}, X_Y^{t-\varepsilon})$, for $\varepsilon > 0$ sufficiently small. In this article, cohomology groups have coefficients in an arbitrary fixed field. If $\{c_0, \dots, c_k\}$ are the critical values of the distance function, then no topology change happens for the set X_Y^a when $a \in (c_j, c_{j+1})$. This is a consequence of the deformation lemma for locally Lipschitz functions.

D.27 **Lemma.** *Let $f\colon X \to \mathbf{R}$ be a proper, locally Lipschitz function. Assume that the interval $[a,b]$ contains no critical values of f. Then X^b deformation retracts to X^a. Moreover, if f is definable, then X^b deformation retracts to X^a even if a is a critical value, as soon as there are no other critical values in $(a,b]$.*

Proof. The first part of the statement follows from Clarke's implicit function theorem [18, Theorem 1], see [1, Proposition 1.2]. The second part follows from the first part and the definable triviality [25, Chapter 9, Theorem 1.2]. □

D.2.5 Passing a critical value: general case

Observe first that, if X, Y are definable with X smooth, when $t \in \mathbf{R}$ passes a critical value, the topology of X_Y^t changes in a controlled way. This follows again from the definable triviality and is an application of the next result.

D.28 **Proposition.** *Let X, Y be definable with X smooth and let $f\colon X \to \mathbf{R}$ be a Lipschitz, definable function. For every critical value $c \in \mathbf{R}$ there exists a definable set $K(c)$ such that for all $\varepsilon > 0$ small enough*

$$H^*(X_Y^{c+\varepsilon}, X_Y^{c-\varepsilon}) \cong \tilde{H}^*(K(c)).$$

Proof. This is clear, since for $\varepsilon > 0$ small enough all the pairs $(X_Y^{c+\varepsilon}, X_Y^{c-\varepsilon})$ are homotopy equivalent (by definable triviality) and therefore one can define $K(c) := X_Y^{c+\varepsilon}/X_Y^{c-\varepsilon}$. Since definable functions can be triangulated, then up to a definable homeomorphism, $X_Y^{c-\varepsilon}$ is a subcomplex of $X_Y^{c+\varepsilon}$ and therefore $K(c)$ is definable. □

D.29 **Remark.** More generally, an analogue of the previous proposition still holds true for a proper definable function $f\colon X \to \mathbf{R}$ (without the assumption that it is locally Lipschitz). In fact, by definable triviality [25, Chapter 9, Theorem 1.2], one can partition $\mathbf{R}$ into finitely many points and intervals such that on each interval f is a trivial fibration. The sublevel sets X^t of f can only change when we pass the points – we don't call these critical points (because f might not be locally Lipschitz), but the relative cohomology is still that of a definable set. ■

D.2.6 Passing a critical value: nondegenerate case

In order to say something more on the change in topology of X_Y^t when t passes a critical value, in a way that is similar to Morse theory, we need to introduce the notion of **nondegenerate** critical points. This notion can be introduced at least for a special class of locally Lipschitz functions called **continuous selections**.

D.30 **Definition** (Continuous selection). Let X be a smooth manifold and $\{f_0, \dots, f_m\}$ a finite subset of $\mathscr{C}^2(X, \mathbf{R})$. We say that $f\colon X \to \mathbf{R}$ is a **continuous selection** from $\{f_0, \dots, f_m\}$ if f is continuous and for every $x \in X$ there exists $i \in \{0, \dots, m\}$ such that $f(x) = f_i(x)$. In this case we will write $f \in \mathrm{CS}(f_0, \dots, f_m)$. ■

If $f \in \mathrm{CS}(f_0, \dots, f_m)$, for $i \in \{0, \dots, m\}$ we denote by $S_i \subseteq X$ the set

$$S_i := \left\{ x \in X \,\middle|\, f(x) = f_i(x) \right\}.$$

Given $x \in X$ the **effective index set** of $f \in \mathrm{CS}(f_0, \dots, f_m)$ at $x \in X$ is defined by

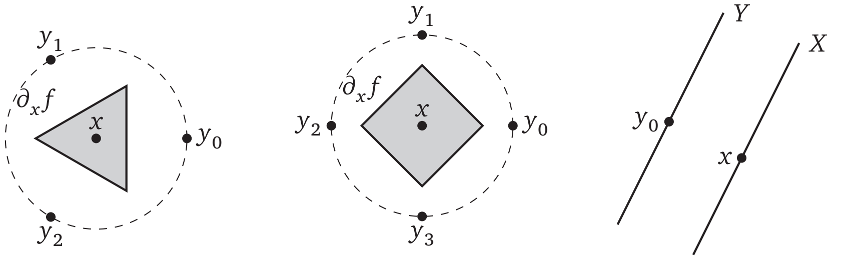
$$I(x) := \left\{ i \in \{0, \dots, m\} \,\middle|\, x \in \overline{\mathrm{int}(S_i)} \right\}.$$

Note that continuous selections are locally Lipschitz [1], so we can talk about their critical points using the definitions above. Again, below we assume that X is endowed with some Riemannian structure, so as to identify its tangent and cotangent bundles.

D.31 **Definition** (Nondegenerate critical points of continuous selections). Let $f \in \mathrm{CS}(f_0, \dots, f_m)$. A critical point $x \in C(f)$ is said to be **nondegenerate** (with respect to $f_0, \dots, f_m$) if the following conditions are satisfied:

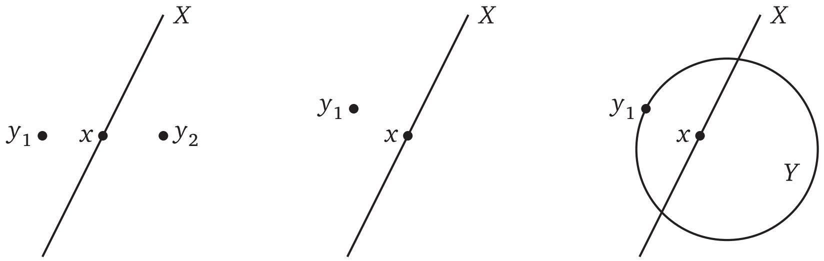
1. For every $i \in I(x)$ the set of differentials $\{D_x f_j \mid j \in I(x) \smallsetminus \{i\}\}$ is linearly independent.
2. Denote by $(\lambda_i)_{i \in I(x)}$ the unique list of nonnegative real numbers such that

$$\sum_{i \in I(x)} \lambda_i = 1 \quad \text{and} \quad \sum_{i \in I(x)} \lambda_i D_x f_i = 0$$



(a) nondegenerate critical point (b) degenerate critical point for the first condition (c) degenerate critical point for the second condition

Figure D.3: Examples of critical points of $f = \mathrm{dist}$ and their degeneracy. (a) and (b): X is the plane $\mathbf{R}^2$ and Y is a finite set of points in the plane. (c): X and Y are lines.



(a) indices $(1,0)$ (b) indices $(0,0)$ (c) indices $(0,1)$

Figure D.4: Examples of dist and $x \in X$ with different piecewise linear and quadratic indices, written as $(k(x), \iota(x))$. (a) and (b): X is a line and Y is a set of points. (c): X is a line, Y is a circle, and y_1 is the closest point of Y to x.

and by $\lambda f \colon X \to \mathbf{R}$ the function defined by $\lambda f(z) := \sum_{i \in I(x)} \lambda_i f_i(z)$. Then the second differential of λf is nondegenerate on

$$V(x) := \bigcap_{i \in I(x)} \ker(D_x f_i).$$

We denote by $k(x) := \#I(x) - 1$ and call this number the **piecewise linear index**, and by $\iota(x)$ the negative inertia index of the restriction of the second differential of λf to $V(x)$. We call $\iota(x)$ the **quadratic index**. See Figure D.3 for an illustration of nondegenerate and degenerate critical points of $\mathrm{dist}_Y|_X$, and Figure D.4 for an illustration of different indices for $\mathrm{dist}_Y|_X$.

Clearly, in the case X is endowed with a Riemannian metric, the above conditions can be rephrased using the gradients of the f_i instead of their differential.

Moreover, the nondegeneracy of the second differential of λf on $V(x)$ is equivalent to the nondegeneracy of the restriction of the corresponding quadratic form on it. ∎

D.32 **Remark.** An important remark concerns the reformulation of these conditions in the case X is a submanifold of another manifold N and we are considering the restriction of a continuous selection $f = g|_X$ where $g \in \mathrm{CS}(g_0, \dots, g_m)$ with $g_i \in \mathscr{C}^2(N, \mathbf{R})$, so that

$$f_j = g_j|_X, \quad j = 0, \dots, m.$$

Suppose that X is given as the zero set of a regular equation $p = 0$, with $p : N \to \mathbf{R}$ a smooth function (in Section D.4 we are interested in the case $N = \mathbf{R}^n$ and p is a polynomial). Then, for all $j = 0, \dots, m$,

$$D_x f_j = D_x g_j|_{T_x X} = D_x g_j|_{\ker(D_x p)}.$$

Rewriting the above conditions with this notation, we see that $x \in X$ is a nondegenerate critical point of $f = g|_X$ if and only if

1. for every $i \in I(x)$ the set of differentials $\{D_x g_j|_{\ker(D_x p)} \mid j \in I(x) \setminus \{i\}\}$ is linearly independent;
2. denote by $\{\mu, (\lambda_i)_{i \in I(x)}\}$ the unique list of numbers such that the λ_i are nonnegative, $\mu \neq 0$,

$$\sum_{i \in I(x)} \lambda_i = 1, \quad \text{and} \quad \mu D_x p + \sum_{i \in I(x)} \lambda_i D_x g_i = 0,$$

   and by $\lambda f \colon X \to \mathbf{R}$ the function defined by $\lambda f(z) := \sum_{i \in I(x)} \lambda_i f_i(z)$. Then the second differential of λf is nondegenerate on

$$V(x) := \bigcap_{i \in I(x)} \ker(D_x f_i|_{\ker(D_x p)}) = \ker(D_x p) \cap \left(\bigcap_{i \in I(x)} \ker(D_x f_i) \right).$$

In order to compute the second differential of λf on $V(x)$ using the original functions, we proceed as follows. We consider the function

$$\widetilde{\lambda g} := \mu p + \sum_{i \in I(x)} \lambda_i g_i \colon N \to \mathbf{R}.$$

Observe that, since $X = \{p = 0\}$, then $\widetilde{\lambda g}|_X = \lambda f$. Moreover, $\widetilde{\lambda g}$ has a critical point at $x \in N$ and therefore the second differential of its restriction to X equals the restriction of its second differential to T_xX. This means that the condition that the second differential of λf is nondegenerate on $V(x)$ is equivalent to the condition that

$$\mu D_x^2 p + \sum_{i\in I(x)} \lambda_i D_x^2 g_i \quad \text{is nondegenerate on } V(x). \qquad \blacksquare$$

D.33 **Remark.** Continuous selections have normal forms at their critical points, in a fashion similar to the Morse Lemma [44, Lemma 2.2]. More precisely, if x is a nondegenerate critical point of $f \in \mathrm{CS}(f_0,\dots,f_m)$ with piecewise linear index $k = k(x)$ and quadratic index $\iota = \iota(x)$, then there exists a neighborhood U of x and a locally Lipschitz homeomorphism $\psi\colon \mathbf{R}^k \times \mathbf{R}^{n-k} \to U$ such that

$$f(\psi(t_1,\dots,t_k,t_{k+1},\dots,t_n)) = f(x) + \ell(t_1,\dots,t_k) - \sum_{j=k+1}^{k+\iota} t_j^2 + \sum_{j=k+\iota+1}^{n} t_j^2, \quad \text{(D.5)}$$

where $\ell(t) \in \mathrm{CS}(t_1,\dots,t_k,-(t_1+\dots+t_k))$, see [36, Theorem 3.3]. Note in particular that nondegenerate critical points of continuous selections are isolated. $\blacksquare$

In the special case of a continuous selection of the form $f = \min\{f_0,\dots,f_m\}$, if x is a nondegenerate critical point with indices $k = k(x)$ and $\iota = \iota(x)$, then the piecewise linear continuous selection ℓ in the normal form (D.5) is

$$\ell(t_1,\dots,t_k) = \min\{t_1,\dots,t_k,-(t_1+\dots+t_k)\},$$

see [36, Theorem 3.3]. In particular, in this case, if f is also definable, then by [1, Sect. 4] it follows the set $K(c)$ defined in Proposition D.28 depends only on k and ι.

D.34 **Proposition.** *Let $f = \min\{f_0,\dots,f_m\}$ be a continuous definable selection. Assume that all the critical points of f at level $c \in \mathbf{R}$ are nondegenerate and denote them by $\{x_1,\dots,x_\nu\}$. Then the set $K(c)$ from Proposition D.28 equals*

$$K(c) = \bigsqcup_{j=1}^{\nu} S^{k(x_j)+\iota(x_j)}.$$

Proof. This follows from [1, Sect. 4], analyzing the conditions of [1, Theorem 4.2] case by case. $\square$

We will see in Proposition D.36 a sufficient condition for the distance function dist_Y being of the form $\min\{f_0,\dots,f_m\}$ in a neighborhood of a point x.

D.35 **Remark.** Let x be a nondegenerate critical point with indices $k = k(x)$ and $\iota = \iota(x)$ of a continuous selection $f = \min\{f_0, \dots, f_m\}$. Then f is topologically equivalent to a quadratic form [36, Corollary 3.4]: there exists a neighborhood U of x and a homeomorphism $\psi\colon \mathbf{R}^n \to U$ such that

$$f(\psi(t_1, \dots, t_n)) = f(x) - \sum_{j=1}^{k+\iota} t_j^2 + \sum_{j=k+\iota+1}^{n} t_j^2.$$

(Note that the conclusions of Proposition D.34 can also be derived from this statement, following the standard argument from Morse theory.) We observe that a normal form of f of this type can be also obtained using the theory of Min-type functions, see [31, Sect. 3.2]. ■

D.2.7 Nondegenerate critical points of distance functions

In order to use the notion of nondegenerate critical point for $\mathrm{dist}_Y|_X$ introduced above, we need to make this function a continuous selection of *finitely many* $\mathscr{C}^2$ functions, at least nearby a critical point. In the case Y is smooth, this is done as follows.

Denote by $\varphi\colon NY \to \mathbf{R}^n$ the restriction of the exponential map of $\mathbf{R}^n$ to the normal bundle of Y. Note that we can think of elements in NY as pairs $(y, v) \in \mathbf{R}^{2n}$ with $y \in Y$ and $v \in N_y Y$ and $\varphi(y, v) = y + v$.

D.36 **Proposition.** *Suppose that Y is smooth and let $x \in \mathbf{R}^n$ be a regular value of $\varphi\colon NY \to \mathbf{R}^n$. Then*

1. *$B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set $\{y_0, \dots, y_k\}$,*

and for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

2. *for every $i \in \{0, \dots, k\}$, the function $f_i := \mathrm{dist}_{B(y_i,\varepsilon)\cap Y}$ is $\mathscr{C}^2$ on $B(x, \delta)$,*
3. *the function $\mathrm{dist}_Y|_{B(x,\delta)}$ is a continuous selection of the f_i: specifically, $\mathrm{dist}_Y|_{B(x,\delta)} = \min_i f_i$.*

Proof. Let $y \in B(x, \mathrm{dist}_Y(x)) \cap Y$. Then $(y, x-y) \in NY$ and $\varphi(y, x-y) = x$. Since x is a regular value of φ, no point (y', v') near $(y, x-y)$ can have image x by φ. In particular, this means that y is an isolated point in $B(x, \mathrm{dist}_Y(x)) \cap Y$. Since the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is compact, we deduce that $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set, which we write as $\{y_0, \dots, y_k\}$. This proves (1).

Since x is a regular value of φ, we have, for all $i \in \{0, \ldots, k\}$, neighborhoods U_i of x in $\mathbf{R}^n$, V_i of y_i in Y, and W_i of $x - y_i$ in $\mathbf{R}^{n-\dim Y} \cong N_{y_i} Y$ such that $V_i \times W_i \subseteq NY$ (up to trivialization) and $\varphi_i := \varphi|_{V_i \times W_i} : V_i \times W_i \to U_i$ is a diffeomorphism. In particular, we can choose these neighborhoods such that every point of U_i is a regular value of φ and such that $f = \min_i \operatorname{dist}_{V_i}$ on $U := \bigcap_i U_i$. We define the function $\psi_i : U_i \to V_i$ by $\psi_i(z) := \operatorname{proj}_1(\varphi_i^{-1}(z))$. In particular, $\psi_i(x) = y_i$. Note that ψ_i is a $\mathscr{C}^2$ function. Then, for all z in U_i, $\operatorname{dist}_{V_i}(z) = \|\psi_i(z) - z\|$, and so the function $\operatorname{dist}_{V_i}$ is also $\mathscr{C}^2$.

For the choice of V_i, we can choose a ball around y_i with radius $\varepsilon_i > 0$, and set $\varepsilon = \min_i \varepsilon_i$. This restricts the choice of the U_i, but we can always take δ such that $B(x, \delta) \subseteq U$. Then the functions $f_i := \operatorname{dist}_{V_i}$ satisfy (2) and (3). □

Related results to Proposition D.36 can be found in [51, Sect. 3].

D.37 **Example.** The conclusions of Proposition D.36 are not valid if we drop the assumption that x is a regular value of the normal exponential map. For instance, in the case of a smooth parabola $Y = \{x_2 = x_1^2\}$ in $\mathbf{R}^2$, the set M_Y equals the half line $\{(0,t), t > 1\}$. The point $x = (0,1) \in \overline{M_Y}$ has a unique closest point in Y (the origin), but it is not a regular value of the normal exponential map. Moreover every point x' of the form $x' = (0,t)$ with $t > 1$ has two closest points in Y and so the function dist_Y is not differentiable at x, and therefore it cannot be $\mathscr{C}^2$, contradicting Proposition D.36(2). ■

D.2.8 Morse theory of distance functions

Let us go back now to the problem of understanding the contribution of a nondegenerate critical point of $\operatorname{dist}_Y|_X$ to the homology of X. To start with, we recall the following result from semialgebraic geometry, see [41, Theorem 1.42]. The result shows that compact definable sets have definable mapping cylinder neighborhoods (the proof therein works in the definable framework).

D.38 **Theorem.** *Let S be a compact definable set and $f : S \to [0, \infty)$ be a continuous definable function. Denote by $Z := f^{-1}(0)$ the zero set of f. Then there exists $\varepsilon_f > 0$ such that for every $\varepsilon \in (0, \varepsilon_f)$ the set $\{f \leq \varepsilon\}$ is a mapping cylinder neighborhood of Z in S. In particular, the inclusions $Z \hookrightarrow \{f < \varepsilon\} \hookrightarrow \{f \leq \varepsilon\}$ are homotopy equivalences.*

Going back to our setting, note that $X \cap Y$ is always critical and we have the following corollary.

D.39 **Corollary.** *Let Y be closed and definable and X smooth and definable in $\mathbf{R}^n$. Then, for all $R > 0$, there exists $\varepsilon > 0$ such that, for all $t \in (0, \varepsilon)$, $X_Y^t \cap B(0, R)$ deformation retracts to $X_Y^0 \cap B(0, R)$.*

Proof. Given $R > 0$ consider the compact definable set $S := X \cap B(0, R)$ together with the continuous definable function $f \colon S \to [0, \infty)$ given by $\alpha := \mathrm{dist}_Y|_S$. Then the conclusion follows from the definable version of Theorem D.38. □

The following definition is useful.

D.40 **Definition.** Let $Y \subseteq \mathbf{R}^n$ be closed and definable and X be a smooth manifold. We say that a point x of X is a **nondegenerate critical point of** $\mathrm{dist}_Y|_X$ if $\mathrm{dist}_Y|_X$ is a continuous selection at x and x is a nondegenerate critical point for the continuous selection. For every $k, \iota \in \mathbf{N}$ we denote by $C_{k,\iota}(X, Y)$ the set of nondegenerate critical points of $\mathrm{dist}_Y|_X$ with piecewise linear index k and quadratic index ι and which are not on $X \cap Y$. ■

The next result is a specialization of Propositions D.28 and D.34 for nondegenerate critical points of distance functions.

D.41 **Corollary.** *Let $Y \subseteq \mathbf{R}^n$ be closed and definable and $X \subseteq \mathbf{R}^n$ be a smooth manifold. Let $c \neq 0$ be a critical value of $\mathrm{dist}_Y|_X$ such that the set*

$$C(\mathrm{dist}_Y|_X) \cap (\mathrm{dist}_Y|_X)^{-1}(c) = \{x_1, \ldots, x_\nu\}$$

consists only of nondegenerate critical points. For every $j = 1, \ldots, \nu$ denote by (k_j, ι_j) the pair of numbers such that $x_j \in C_{k_j,\iota_j}(X, Y)$. Then for $\varepsilon > 0$ sufficiently small,

$$H^*(X_Y^{c+\varepsilon}, X_Y^{c-\varepsilon}) \cong \bigoplus_{j=1}^{\nu} \tilde{H}^*(S^{k_j+\iota_j}).$$

D.42 **Definition** (Nondegenerate pairs). We say that a pair (X, Y) is a **nondegenerate pair** if all critical points of $\mathrm{dist}_Y|_X$ not on $X \cap Y$ are nondegenerate. ■

For a pair of spaces $Z' \subseteq Z$, let $b_i(Z)$ denote the dimension of $H^i(Z)$ as a vector space over the fixed field of coefficients, and let $b_i(Z, Z')$ denote the dimension of $H^i(Z, Z')$. In our setting, strong Morse inequalities can be stated as follows.

D.43 **Proposition** (strong Morse inequalities). *Let $Y \subseteq \mathbf{R}^n$ be a closed and definable set and $X \subseteq \mathbf{R}^n$ be a smooth, compact and definable manifold such that (X,Y) is a nondegenerate pair. Then, for any integer $\lambda \geq 0$,*

$$\sum_{i=0}^{\lambda}(-1)^{i+\lambda}b_i(X) \leq \sum_{i=0}^{\lambda}(-1)^{i+\lambda}\left(b_i(X\cap Y)+\sum_{k+\iota=i}\#C_{k,\iota}(X,Y)\right).$$

Proof. Let $0=c_0<c_1<\cdots<c_m$ denote the critical values of $\mathrm{dist}_Y|_X$, which are finitely many by Proposition D.25. Choose real numbers $\{d_j\}_{j=0}^{m+1}$ with $d_0<c_0$, $c_m<d_{m+1}$, and $c_{j-1}<d_j<c_j$ for all $j\in\{1,\dots,m\}$. Proceeding exactly as in [44, Sect. 5] one obtains, for every integer $\lambda \geq 0$, the inequality

$$\sum_{i=0}^{\lambda}(-1)^{i+\lambda}b_i(X_Y^{d_{m+1}}) \leq \sum_{j=0}^{m}\sum_{i=0}^{\lambda}(-1)^{i+\lambda}b_i(X_Y^{d_{j+1}},X_Y^{d_j}).$$

On the left-hand side, we observe that $b_i(X_Y^{d_{m+1}})=b_i(X)$. On the right-hand side, we can rearrange the sum as

$$\sum_{i=0}^{\lambda}(-1)^{i+\lambda}\left(\sum_{j=0}^{m}b_i(X_Y^{d_{j+1}},X_Y^{d_j})\right)$$

and observe that $\sum_{j=0}^{m}b_i(X_Y^{d_{j+1}},X_Y^{d_j})=b_i(X\cap Y)+\sum_{k+\iota=i}\#C_{k,\iota}(X,Y)$ by Corollary D.41, noting that the term $b_i(X\cap Y)$ comes from crossing the 0^{th} critical value. □

The weak Morse inequalities

$$b_\lambda(X) \leq b_\lambda(X\cap Y)+\sum_{k+\iota=\lambda}\#C_{k,\iota}(X,Y)$$

are obtained from Proposition D.43 by adding the strong inequalities for λ and $\lambda-1$. Writing $\chi(Z) := \sum_{i\geq 0}(-1)^i b_i(Z)$, we obtain the following interesting result that relates the Euler characteristics of X and Y to the critical points of the corresponding relative distance functions.

D.44 **Corollary.** *Let $Y \subseteq \mathbf{R}^n$ be a closed and definable set and $X \subseteq \mathbf{R}^n$ be a smooth, compact, and definable manifold such that (X,Y) is a nondegenerate pair. Then*

$$\chi(X\cap Y)+\sum_{k,\iota\geq 0}(-1)^{k+\iota}\#C_{k,\iota}(X,Y)=\chi(X).$$

Furthermore, if Y is smooth, X is closed and definable, and (Y,X) is a nondegenerate pair, we get the following equation:

$$\chi(Y)+\sum_{k,\iota\geq 0}(-1)^{k+\iota}\#C_{k,\iota}(X,Y)=\chi(X)+\sum_{k,\iota\geq 0}(-1)^{k+\iota}\#C_{k,\iota}(Y,X). \tag{D.6}$$

Proof. As in the classical case, the first equality follows from Proposition D.43 by comparing the strong Morse inequalities for large enough values λ and $\lambda+1$. The second equality follows from the first one applied to the pairs (X,Y) and (Y,X). □

D.45 **Remark.** The conclusions of Proposition D.43 are still valid if $X=\mathbf{R}^n$ and Y is compact, since this guarantees that the distance function from Y is proper. ■

D.46 **Example.** Let $Y=\{y\}$ with $y\notin X$. Then the piecewise linear index of $\mathrm{dist}_Y|_X$ at each critical point is zero, and the quadratic index is the standard Morse index, and so the left-hand side of (D.6) equals $1+\sum_\iota(-1)^\iota\#C_{0,\iota}(\mathrm{dist}_{\{y\}}\mid_X)$. On the other hand, $\mathrm{dist}_X\mid_{\{y\}}$ is a constant function, with critical point $\{y\}$ of total index zero, and so the right hand side equals $\chi(X)+1$. In this case, this formula gives back the Euler Characteristic of X as the alternating sum of the number of critical points of quadratic index ι. ■

D.47 **Example** (Distance function from a finite set, continuation of Example D.17). In this example we provide further details on nondegenerate critical points of dist_Y, with $Y=\{y_0,\dots,y_m\}\subset\mathbf{R}^n$, and review abstract and geometric complexes capturing the topology of sublevels sets of dist_Y.

For the generic choice of the points $\{y_0,\dots,y_m\}$, all critical points of dist_Y are nondegenerate and the critical points can be found with the procedure described in Example D.17. In view of Definition D.31 and Section D.1.5, we see that the piecewise linear index k of a critical point is not an analogue in the context of distance functions of the classical Morse index; in this case, however, as the quadratic index ι is always zero, the piecewise linear index is what determines the contribution to the topology change. More precisely, using Corollary D.41, we see that if $c\neq 0$ is a critical value of dist_Y such that $C(\mathrm{dist}_Y)\cap\mathrm{dist}_Y|_X^{-1}(c)=\{x_1,\dots,x_\nu\}$ consists only of nondegenerate critical points, then for $\varepsilon>0$ sufficiently small,

$$H^*((\mathbf{R}^n)_Y^{c+\varepsilon},(\mathbf{R}^n)_Y^{c-\varepsilon})\cong\bigoplus_{j=1}^{\nu}\tilde H^*(S^{k_j}).$$

The critical points of dist_Y can be characterized via the interplay of the Voronoi diagram and the Delaunay tessellation, see e.g. [14]. The **Delaunay tessellation** of a finite set $Y \subset \mathbf{R}^n$ is the dual of the Voronoi diagram of Y. It is a polytopal complex (i.e., a polyhedral complex such that all cells are bounded) embedded in $\mathbf{R}^n$ and covering the convex hull of Y. If Y is in general position, it is furthemore a (geometric) simplicial complex. The set of vertices of the Delaunay tessellation is Y. An n-dimensional Delaunay cell is the convex hull of $n+1$ or more points in Y if the intersection of their corresponding Voronoi cells is a vertex of the Voronoi diagram. Delaunay cells and their faces constitute the Delaunay tessellation. If x is a nondegenerate critical point of dist_Y, then the convex hull $\mathrm{co}(B(x, \mathrm{dist}_Y) \cap Y)$ is the Delaunay face dual to the lowest dimensional Voronoi face containing x. In the rest of this example, let us assume Y to be in general position and all critical points of dist_Y to be nondegenerate. Then, the critical points of dist_Y are exactly the intersection between the Voronoi faces and their dual Delaunay faces [14].

For $n = 2$, the topology of the sublevels sets of the distance function is described in details in [50]. For an arbitrary n, some abstract and geometric complexes are known to have the same homotopy type of the sublevel sets of dist_Y.

The **Čech complex** of Y at scale t, denoted by $\check{C}_t(Y)$, is the abstract simplicial compchecklex defined as the nerve of the collection of balls $\{B(y_i, t)\}_{i=0}^m$ in $\mathbf{R}^n$. By the nerve lemma [11, 6], $\check{C}_t(Y)$ is homotopy equivalent to the sublevel set $(\mathbf{R}^n)_Y^t = \bigcup_{i=0}^m B(y_i, t)$ of dist_Y.

The subcomplex $\mathrm{Del}_t(Y)$ of $\check{C}_t(Y)$ given by the nerve of the collection $\{B(y_i, t) \cap V(y_i)\}_{i=0}^m$ of balls intersected with the respective Voronoi cells is called **α-shape** or **Delaunay complex** of Y at scale t [26]. If Y is in general position, $\mathrm{Del}_t(Y)$ is embedded in $\mathbf{R}^n$ as a subcomplex of the Delaunay tessellation, which is a simplicial complex in $\mathbf{R}^n$ in this case. The Čech and Delaunay complexes at scale t are homotopy equivalent [26], and the latter can be obtained from the former via a sequence of simplicial collapses [5].

The **flow complex** of Y [32, 14] is a polytopal complex in $\mathbf{R}^n$ defined using the critical points of dist_Y. A vector field on $\mathbf{R}^n$ can be defined using Voronoi faces and their dual Delaunay faces. The critical points of dist_Y are exactly the fixed points of a flow associated with the vector field. The stable manifolds of the critical points of dist_Y with respect to the flow define the flow complex, see [14, Sect. 3]; note that here we consider an immediate adaptation of the definition, using the function dist_Y instead of dist_Y^2. The flow complex has exactly one cell for each critical point of dist_Y.

Polytopal subcomplexes of the flow complex can be defined for each scale t. The underlying spaces of these complexes are called **flow shapes** of Y at scale t, denoted $\mathrm{F}_t(Y)$, and are obtained by applying the flow complex construction only to critical points x such that $\mathrm{dist}_Y(x) \leq t$. The flow shape $\mathrm{F}_t(Y)$ is homotopy equivalent [14] to the sublevel set $(\mathbf{R}^n)_Y^t = \bigcup_{i=0}^m B(y_i, t)$ of dist_Y, hence also to $\check{\mathrm{C}}_t(Y)$ and $\mathrm{Del}_t(Y)$. Considering all scales t, by Corollary D.41 we know that the filtration of flow shapes $\{\mathrm{F}_t(Y)\}_{t\geq 0}$ realizes the minimum number of cells necessary to have these homotopy equivalences at each scale. ■

D.3 Examples

D.3.1 Distance from a hypersurface

Let Y be a smooth compact hypersurface in $\mathbf{R}^n$, and let $x \in \mathbf{R}^n \setminus Y$ be a non-degenerate critical point of dist_Y. Let us suppose that x is a regular value of the normal exponential map $\varphi\colon NY \to \mathbf{R}^n$, so that we are in the situation of Proposition D.36, and let $\{y_0, \dots, y_k\}$ be the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ of closest points to x on Y.

For $\varepsilon > 0$ small enough as in Proposition D.36, let $g_{y_i} := \mathrm{dist}_{B(y_i,\varepsilon)\cap Y}$. Then, in a neighborhood of x the function dist_Y is a continuous selection of $\mathscr{C}^2$ functions, of the form $\min_{i\in\{0,\dots,k\}} g_{y_i}$. Letting

$$v_i := \nabla g_{y_i}(x) = \frac{x - y_i}{\|x - y_i\|} = \frac{x - y_i}{\mathrm{dist}_Y(x)}, \tag{D.7}$$

we denote by $(\lambda_i)_{i\in\{0,\dots,k\}}$ the unique list of nonnegative numbers (see Definition D.31) such that $\sum_{i=0}^k \lambda_i = 1$ and $\sum_{i=0}^k \lambda_i v_i = 0$.

We describe some properties of the Hessian matrices $H(g_{y_i})$ at x, using [2, Theorem 3.2] (see also [43, Section 3]). For every fixed $i \in \{0, \dots, k\}$, there exists an orthonormal basis $\{e_{i,1}, \dots, e_{i,n}\}$ of $\mathbf{R}^n$ such that $N_{y_i}Y = \mathbf{R}\{e_{i,n}\}$ in which the matrix $H(g_{y_i})(x)$ is diagonal. The eigenvalue $\beta_{i,n}$ corresponding to the eigenvector $e_{i,n}$ is zero, while the remaining eigenvalues $\beta_{i,1}, \dots, \beta_{i,n-1}$ are nonzero since x is nondegenerate, and are given by the formula

$$\beta_{i,j} = \frac{-\kappa_{i,j}}{1-\mathrm{dist}_Y(x)\kappa_{i,j}}, \quad j = 1,\ldots,n-1, \tag{D.8}$$

where $\kappa_{i,j}$ are the principal curvatures at y_i, i.e. the eigenvalues of the shape operator at $y_i \in Y$, see [40, Chapter 8], defined using the *inner* unit normal at y_i.

We now study the quadratic index $\iota(x)$ of dist_Y at x, which is the negative inertia index of the convex combination $\sum_{i=0}^{k} \lambda_i H(g_{y_i})$ (with coefficients λ_i defined as above) restricted to the $(n-k)$-dimensional subspace $V(x) = \mathrm{Span}\{v_0,\ldots,v_k\}^\perp$ of $\mathbf{R}^n$, see Definition D.31. As a consequence of the properties we just discussed, one can state sufficient conditions for determining the quadratic index $\iota(x)$ based on the simple fact that a convex combination of positive (respectively, negative) semidefinite matrices is positive (resp., negative) semidefinite.

D.48 **Corollary.** *In the situation we are considering in Section D.3.1,*

- *if $\beta_{i,j} > 0$ for all $i \in \{0,\ldots,k\}$ and $j \in \{1,\ldots,n-1\}$, then $\iota(x) = 0$;*
- *if $\beta_{i,j} < 0$ for all $i \in \{0,\ldots,k\}$ and $j \in \{1,\ldots,n-1\}$, then $\iota(x)$ takes the maximum value $n-k$.*

If not all the eigenvalues $\beta_{i,j}$ have the same sign, determining $\iota(x)$ is more complicated. A general expression for $\iota(x)$ in terms of the eigenvalues $\beta_{i,j}$ is out of reach, since in general the matrices $H(g_{y_i})(x)$ cannot be diagonalized in a common basis. However, this is possible in the case of $Y \subset \mathbf{R}^2$, which we now describe more in detail.

D.3.2 Distance from a curve in the plane

We now focus on the case $n = 2$, with Y a smooth compact curve in $\mathbf{R}^2$. As in Section D.3.1, let $x \in \mathbf{R}^2 \setminus Y$ be a nondegenerate critical point of dist_Y which is a regular value of the normal exponential map. By nondegeneracy of x, the number of its closest points on Y is $2 \le \#(B(x,\mathrm{dist}_Y(x)) \cap Y) \le 3$.

If $B(x,\mathrm{dist}_Y(x)) \cap Y = \{y_0,y_1,y_2\}$, then the piecewise linear index is $k(x) = 2$ and the quadratic index is $\iota(x) = 0$ since $V(x) = \mathrm{Span}\{v_0,v_1,v_2\}^\perp = 0$. Let us consider the case where $B(x,\mathrm{dist}_Y(x)) \cap Y = \{y_0,y_1\}$, which means $k(x) = 1$. As described in Section D.3.1, for each $i \in \{0,1\}$ the exists an orthonormal basis

$\{e_{i,1}, e_{i,2}\}$ of $\mathbf{R}^2$ such that $N_{y_i}Y = \mathbf{R}\{e_{i,2}\}$ in which $H(g_{y_i})(x)$ is diagonal. In this case, consider v_0 and v_1 defined as in Equation (D.7), and observe that they are parallel since $x = \frac{1}{2}(y_0 + y_1)$. We have $N_{y_0}Y = N_{y_1}Y = \mathrm{Span}\{v_0, v_1\} \cong \mathbf{R}$, and $T_{y_0}Y = T_{y_1}Y = V(x) = \mathrm{Span}\{v_0, v_1\}^{\perp}$. We can diagonalize $H(g_{y_0})(x)$ and $H(g_{y_0})(x)$ in the same basis, obtaining the matrices

$$D_0 = \begin{pmatrix} \beta_{0,1} & 0 \\ 0 & 0 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} \beta_{1,1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $\frac{1}{2}v_0 + \frac{1}{2}v_1 = 0$, the quadratic index $\iota(x)$ is the negative inertia index of the convex combination $\frac{1}{2}D_0 + \frac{1}{2}D_1$ restricted to $V(x)$, which is determined by the sign of $\beta_{0,1} + \beta_{1,1}$:

$$\iota(x) = \begin{cases} 0 & \text{if } \beta_{0,1} + \beta_{1,1} > 0 \\ 1 & \text{if } \beta_{0,1} + \beta_{1,1} < 0. \end{cases}$$

Note that this refines Corollary D.48. The case $\beta_{0,1} + \beta_{1,1} = 0$ does not happen, as it would violate the nondegeneracy of x.

Via an easy calculation, one can see how the principal curvatures determine the sign of $\beta_{0,1} + \beta_{1,1}$, and hence $\iota(x)$. To simplify notations, here we drop the index j, denoting $\beta_i := \beta_{i,1}$ and $\kappa_i := \kappa_{i,1}$ for $i \in \{0,1\}$. We also denote $d := \mathrm{dist}_Y(x)$.

First we observe that, by [44, §6], $\kappa_i < \frac{1}{d}$, for $i \in \{0,1\}$, which is equivalent to $1 - d\kappa_i > 0$. As we saw, $\iota(x)$ is determined by the sign of

$$\beta_0 + \beta_1 = \frac{-\kappa_0}{1 - d\kappa_0} + \frac{-\kappa_1}{1 - d\kappa_1} = \frac{2d\kappa_0\kappa_1 - \kappa_0 - \kappa_1}{(1 - d\kappa_0)(1 - d\kappa_1)}. \tag{D.9}$$

The first equality immediately shows that $\iota(x) = 0$ if $\kappa_0, \kappa_1 < 0$, and that $\iota(x) = 1$ if $\kappa_0, \kappa_1 > 0$. Since the denominator of the right-hand side is positive, the sign of $\beta_0 + \beta_1$ is the same as the sign of the numerator

$$h(\kappa_0, \kappa_1) := 2d\kappa_0\kappa_1 - \kappa_0 - \kappa_1 = 2d\left[\left(\kappa_0 - \frac{1}{2d}\right)\left(\kappa_1 - \frac{1}{2d}\right) - \frac{1}{4d^2}\right],$$

which is easy to study.

The following statement summarizes the possible values of $k(x)$ and $\iota(x)$.

D.49 **Corollary.** *Let $Y \subset \mathbf{R}^2$ be a smooth compact curve, and let $x \in \mathbf{R}^2 \smallsetminus Y$ be a nondegenerate critical point of* dist_Y *which is a regular value of the normal* *exponential map $\varphi : NY \to \mathbf{R}^2$. Then, the piecewise linear index $k(x)$ and the quadratic index $\iota(x)$ (see Definition D.31) can be determined as follows:*

- *if $B(x, \mathrm{dist}_Y(x)) \cap Y = \{y_0, y_1, y_2\}$, then $k(x) = 2$ and $\iota(x) = 0$;*
- *if $B(x, \mathrm{dist}_Y(x)) \cap Y = \{y_0, y_1\}$, then $k(x) = 1$, and $\iota(x)$ is determined by the principal curvatures κ_i of Y at y_i, for $i \in \{0, 1\}$:*

$$\iota(x) = \begin{cases} 0 & \text{if } \left(\kappa_0 - \frac{1}{2d}\right)\left(\kappa_1 - \frac{1}{2d}\right) > \frac{1}{4d^2} \\ 1 & \text{if } \left(\kappa_0 - \frac{1}{2d}\right)\left(\kappa_1 - \frac{1}{2d}\right) < \frac{1}{4d^2}. \end{cases}$$

D.3.3 The index of a critical point for the distance from a point

In this section we are going to prove the following result, which is useful when one wants to compute the index of a critical point for the function $\mathrm{dist}_Y|_X$ when $Y = \{y\} \not\subseteq X$ (in this case the piecewise linear index is zero, and we are back to the standard setting of Morse theory).

D.50 **Proposition.** *Let $X = Z(p)$ be a smooth hypersurface and $Y = \{y\} \in \mathbf{R}^n$ be a generic point. Let also $x \in X$ be a critical point of $\mathrm{dist}_Y|_X$ (nondegenerate by genericity of y). Denote by $Q_x \in \mathrm{Sym}(n-1, \mathbf{R})$ the matrix representing the second fundamental form of $X \hookrightarrow \mathbf{R}^n$ in the direction of $y - x$ with respect to an orthonormal basis of T_xX. Then*

$$\iota(x) = \#\left\{ \text{eigenvalues of } Q_x \text{ that are strictly larger than } \frac{1}{\|x-y\|} \right\}.$$

Proof. We are going to use the discussion from Remark D.32, which we recall in this context. We have a manifold X described by the regular equation $p = 0$ and we want to find the critical points of the function $f := \mathrm{dist}_{\{y\}}|_X$, together with their quadratic index. Notice that in this case f is smooth (because we are assuming $y \notin X$) and the critical points are found using the Lagrange multipliers rule: $x \in X$ is critical if and only if there exists $\lambda \neq 0$ such that

$$\nabla p(x) + \lambda \frac{x-y}{\|x-y\|} = 0.$$

Given such λ (which is unique), the quadratic index of f at x is given by the negative inertia index of

$$\left(\mathrm{H}(p)(x) + \lambda \mathrm{H}(\mathrm{dist}_{\{y\}})(x) \right)\Big|_{T_xX}.$$

Notice that we have the degree of freedom of choosing $-p$ instead of p as a defining polynomial for X and we choose the sign in such a way that $\nabla p(x)$ has the same orientation as $y-x$. Then $\lambda = \|\nabla p(x)\|$.

Moreover, the Hessian of $\mathrm{dist}_{\{y\}}$ can be computed easily: denoting by $w := \frac{x-y}{\|x-y\|}$,

$$\mathrm{H}(\mathrm{dist}_{\{y\}})(x) = \frac{1}{\|x-y\|}\,\mathrm{Id} - ww^\top.$$

In particular, since $w \perp T_xX$, as quadratic forms,

$$\Big(\mathrm{H}(p)(x) + \lambda \mathrm{H}(\mathrm{dist}_{\{y\}})(x)\Big)\Big|_{T_xX} = \Big(\mathrm{H}(p)(x) + \frac{\|\nabla p(x)\|}{\|x-y\|}\,\mathrm{Id}\Big)\Big|_{T_xX}.$$

Recall now that the second fundamental form q_x of $Z(p)$ at x in the direction of $\nabla p(x)$, as a quadratic form, is related to the Hessian of p as follows (this is proved in Lemma D.64 below):

$$\mathrm{H}(p)(x) = -\|\nabla p(x)\| q_x,$$

and therefore, denoting by $\mathrm{i}^+(A)$ and $\mathrm{i}^-(A)$ the number of positive and negative eigenvalues of a symmetric matrix A,

$$\begin{aligned}
\mathrm{i}^-\Big(\mathrm{H}(p)(x) + \frac{\|\nabla p(x)\|}{\|x-y\|}\,\mathrm{Id}\Big)\Big|_{T_xX} &= \mathrm{i}^-\Big(-\|\nabla p(x)\| q_x + \frac{\|\nabla p(x)\|}{\|x-y\|}\,\mathrm{Id}\Big)\Big|_{T_xX} \\
&= \mathrm{i}^-\Big(-q_x + \frac{1}{\|x-y\|}\,\mathrm{Id}\Big)\Big|_{T_xX} \\
&= \mathrm{i}^+\Big(q_x - \frac{1}{\|x-y\|}\,\mathrm{Id}\Big)\Big|_{T_xX}.
\end{aligned}$$

From this last identity the conclusion follows. □

D.4 Genericity in the algebraic setting

Throughout this section, we consider two real n-variable polynomials p and q and two algebraic sets $X := Z(p)$ and $Y := Z(q)$. We assume $n \geq 1$.

D.4.1 A multijet parametric transversality theorem

In this section we prove a useful technical tool, Theorem D.54. We first recall the following classic result, which we adapt from [35, Chapter 3, Theorem 2.7]:

D.51 **Theorem** (Parametric transversality). *Let M, N, P be smooth manifolds, $W \subseteq N$ a submanifold of N, and $F\colon P \times M \to N$ a smooth map. For all p in P, we denote by F_p the map $F(p,-)\colon M \to N$.*

If F is transverse to W, then the set $\{p \in P \mid F(p,-) \pitchfork W\}$ is residual in P. In other words, for generic $p \in P$, the map F_p is transverse to W.

As an immediate consequence we have the following corollary.

D.52 **Corollary.** *Let $F\colon P \times M \to N$ be a submersion and $W \subseteq N$ a submanifold. If $\operatorname{codim}_N W = \dim M$, then for generic $p \in P$ we have $\dim(F(p,M) \cap W) = 0$. If $\operatorname{codim}_N W > \dim M$, then for generic $p \in P$ we have $F(p,M) \cap W = \varnothing$.*

We will be applying these transversality results to **jet spaces**.

D.53 **Definition** (jets and multijets). Let $n, m, r \geq 0$ be integers. The space of **jets of order r from $\mathbf{R}^m$ to $\mathbf{R}^n$** is

$$J^r(\mathbf{R}^m, \mathbf{R}^n) := \mathbf{R}^m \oplus \bigoplus_{\ell=0}^{r} (\mathscr{P}_{\ell,m})^n,$$

where $\mathscr{P}_{\ell,m}$ is the space of homogeneous polynomials of degree ℓ in m variables.

Let $f\colon \mathbf{R}^m \to \mathbf{R}^n$ be a smooth function and $x \in \mathbf{R}^m$. The **jet of order r of f at x** is

$$\begin{aligned} j^r(f)(x) &:= \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right)_{\alpha \in \mathbf{N}^m, |\alpha| \leq r} \in J^r(\mathbf{R}^m, \mathbf{R}^n) \\ &= \left(x, f(x), \ldots, \frac{\partial f}{\partial x_i}(x), \ldots, \frac{\partial^r f}{\partial^r x_i}, \ldots \right). \end{aligned}$$

The identification with polynomial spaces comes from identifying the partial derivative $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)$ with the coefficient in front of the monomial x^α.

Let $k \geq 1$ be an integer. The space of **k-multijets of order r** is just k copies of the space of jets of order r:

$${}_kJ^r(\mathbf{R}^m, \mathbf{R}^n) := J^r(\mathbf{R}^m, \mathbf{R}^n)^k.$$

Similarly, let $f\colon \mathbf{R}^m \to \mathbf{R}^n$ be a smooth function and $\vec{x} := x_1, \ldots, x_k$ distinct points in $\mathbf{R}^m$ (we write $\vec{x} \in \mathbf{R}^{km} \smallsetminus \Delta$). The **$k$-multijet of order r of f at $x_1, \ldots, x_k$** is

$${}_kj^r(\vec{x}) := (j^r(f)(x_i))_{i=1,\ldots,k}.$$

■

We study jets through parametric jet maps of the form

$$F\colon \begin{cases} \mathscr{P}_d \times (\mathbf{R}^{nk} \smallsetminus \Delta) & \to & {}_kJ^r(\mathbf{R}^n, \mathbf{R}) \\ (p, \vec{x}) & \mapsto & {}_kj^r(p)(\vec{x}). \end{cases}$$

The next result is a key technical ingredient for the rest of the paper.

D.54 **Theorem.** *Let $\mathscr{P}_d$ be the space of real polynomials of degree d in $\mathbf{R}^n$. There exists a function $d(n,k,r)$ such that, for all $d \geq d(n,k,r)$, the map $F\colon \mathscr{P}_d \times (\mathbf{R}^{nk} \smallsetminus \Delta) \to {}_kJ^r(\mathbf{R}^n, \mathbf{R})$ is a submersion.*

Proof. Fix d a polynomial degree, $p \in \mathscr{P}_d$ a polynomial, and $\vec{x} := (x_1, \dots, x_k) \in (\mathbf{R}^{nk} \smallsetminus \Delta)$ points in $\mathbf{R}^n$. The differential of F at (p, x) is a linear map

$$DF_{p,\vec{x}}\colon \begin{cases} T_{p,\vec{x}}\mathscr{P}_d \times \mathbf{R}^{nk} & \to & T_{F(p,\vec{x})\,k}J^r(\mathbf{R}^n, \mathbf{R}) \\ (\dot{p}, \dot{\vec{x}}) & \mapsto & \begin{array}{l}(\dot{\vec{x}}, \dots, \dot{p}(x_i) + \nabla p(x_i)^\top \dot{x}_i, \dots, \\ \dots, \nabla \dot{p}(x_i) + \nabla^2 p(x_i)^\top \dot{x}_i, \dots).\end{array} \end{cases}$$

where $T_{p,\vec{x}}\mathscr{P}_d \times \mathbf{R}^{nk} = \mathscr{P}_d \times \mathbf{R}^{nk}$ and $T_{F(p,\vec{x})\,k}J^r(\mathbf{R}^n, \mathbf{R}) = \mathbf{R}^{nk} \oplus \bigoplus_{\ell=0}^r (\mathscr{P}_{\ell,n})^k$.

Let $(\vec{y}, \dot{U}) = (\vec{y}, \vec{u}^{(1)}, \dots, \vec{u}^{(r)}) \in \mathbf{R}^{nk} \oplus \bigoplus_{\ell=1}^r (\mathscr{P}_{\ell,n})^k$. For F to be a submersion at $(p, \vec{x})$, it suffices to find $(\dot{p}, \dot{\vec{x}})$ such that $DF_{p,\vec{x}}(\dot{p}, \dot{\vec{x}}) = (\vec{y}, \dot{U})$. This is linear in the coefficients of $\dot{p}$ and in monomials in $\dot{\vec{x}}$. We find that $\dot{\vec{x}} = \vec{y}$, and then there are $k \sum_{\ell=0}^r \binom{n}{\ell}$ equations in the coefficients of $\dot{p}$. Since the x_i are pairwise distinct points in $\mathbf{R}^n$, these equations are independent. For there to be a solution in general, it suffices that

$$\dim \mathscr{P}_d = \binom{n+d}{d} \geq k \sum_{\ell=0}^r \binom{n}{\ell}.$$

We set $d(n,k,r)$ to be the smallest d satisfying this inequality. $\square$

D.55 **Example.** If $k \leq n+1$ and $r = 1$, then we can take $d(n,k,r) = 3$. Indeed, for all $n \geq 1$, $\binom{n+3}{3} \geq (n+1)^2 \geq k(\binom{n}{0} + \binom{n}{1})$.

If $k \leq n+1$ and $r = 2$, then we can take $d(n,k,r) = 4$. Indeed, we compute

$$\begin{aligned} \dim \mathscr{P}_d - k \sum_{\ell=0}^r \binom{n}{\ell} &\geq \dim \mathscr{P}_d - (n+1) \sum_{\ell=0}^r \binom{n}{\ell} \\ &= \binom{n+4}{4} - (n+1)(\binom{n}{0} + \binom{n}{1} + \binom{n}{2}) \end{aligned}$$

$$\begin{aligned}
&= \frac{(n+1)(n+2)(n+3)(n+4)}{24} - (n+1)(1+n+\frac{n(n-1)}{2}) \\
&= \frac{n+1}{24}((n+2)(n+3)(n+4) - 12(n^2+n+2) \\
&= \frac{n+1}{24}(n^3-3n^2+14n). \qquad \text{(D.10)}
\end{aligned}$$

We determine the sign of (D.10) by looking at the derivative of the last factor: $3n^2-6n+14$ has discriminant $36-4\cdot 3\cdot 14<0$ and value $14>0$ at $n=0$, so the derivative is always positive. At $n=1$, we have $n^3-3n^2+14n=12$, so for all $n\geq 1$, (D.10) is positive. ■

D.4.2 Finite number of closest points

In this section we prove an analogue of [54, Proposition 3], showing that generically the function m_Y is bounded by $n+1$.

D.56 **Theorem** (cf. [54, Proposition 3]). *For the generic $q\in\mathscr{P}_d$ with $d\geq 2$, writing $Y=Z(q)$, for every $x\in\mathbf{R}^n$ the set $B(x,\mathrm{dist}_Y(x))\cap Y$ is a nondegenerate simplex with at most $n+1$ points.*

Proof. For all $k\in\{0,\ldots,n+1\}$, define the parameterized jet map

$$F_k\colon \begin{cases} \mathscr{P}_d\times\mathbf{R}^n\times(\mathbf{R}^{n(k+1)}\smallsetminus\Delta) & \longrightarrow \quad {}_{k+1}J^0(\mathbf{R}^n,\mathbf{R}^2) \\ (q,x,\vec{y}) & \longmapsto \quad (\vec{y},\ldots,q(y_i),\ldots,\|x-y_i\|^2,\ldots) \end{cases}$$

induced by the function $y\mapsto(q(y),\|x-y\|^2)$. Its differential at a point $(q,x,\vec{y})$ is

$$DF_{k,(q,x,\vec{y})}\colon \begin{cases} \mathscr{P}_d\times\mathbf{R}^n\times\mathbf{R}^{n(k+1)} & \longrightarrow \quad \mathbf{R}^{n(k+1)}\times\mathbf{R}^{k+1}\times\mathbf{R}^{k+1} \\ (\dot{q},\dot{x},\dot{\vec{y}}) & \longmapsto \quad \begin{array}{c}(\dot{\vec{y}},\ldots,\dot{q}(y_i)+\nabla q(y_i)^\top\dot{y}_i,\ldots, \\ \ldots,2(x-y_i)^\top(\dot{x}-\dot{y}_i),\ldots).\end{array} \end{cases}$$

Let

$$W_k := \left\{ (\vec{z},\vec{s},\vec{t})\in{}_{k+1}J^0(\mathbf{R}^n,\mathbf{R}^2) \;\middle|\; \begin{array}{l}\forall i\in\{0,\ldots,k\},\ s_i=0, \\ \forall i\in\{1,\ldots,k\},\ t_i=t_0\end{array} \right\}.$$

be a subspace of ${}_{k+1}J^0(\mathbf{R}^n,\mathbf{R}^2)$ and let us prove that F_k is transverse to W_k. Let $(q,x,\vec{y})$ be a point in $\mathscr{P}_d\times\mathbf{R}^n\times(\mathbf{R}^{n(k+1)}\smallsetminus\Delta)$ such that $F_k(q,x,\vec{y})\in W_k$, and $(\dot{\vec{z}},\dot{\vec{s}},\dot{\vec{t}})$ a tangent vector in $T_{F_k(q,x,\vec{y})\,k+1}J^0(\mathbf{R}^n,\mathbf{R}^2)$. We are looking for $(\dot{q},\dot{x},\dot{\vec{y}})\in T_{(q,x,\vec{y})}(\mathscr{P}_d\times\mathbf{R}^n\times(\mathbf{R}^{n(k+1)}\smallsetminus\Delta))$ and $(\vec{w},0,0)\in T_{F_k(q,x,\vec{y})}W_k$ such that

$$(\dot{\vec{z}},\dot{\vec{s}},\dot{\vec{t}}) = DF_{k,(q,x,\vec{y})}(\dot{q},\dot{x},\dot{\vec{y}})+(\vec{w},0,0)$$

$$= (\vec{\dot{y}} + \vec{\dot{w}}, \dots, \dot{q}(y_i) + \nabla q(y_i)^\top \dot{y}_i, \dots, 2(x-y_i)^\top(\dot{x}-\dot{y}_i), \dots).$$

Firstly, we observe that x is different from all the y_i. Indeed, if not, then we would have $\|x-y_i\|^2 = 0$ for all $i \in \{0,\dots,k\}$, which contradicts the fact that the y_i are distinct. So we can set $\dot{x} = 0$ and solve the equations $2(x-y_i)^\top(\dot{x}-\dot{y}_i) = \dot{t}_i$ to obtain the $\dot{y}_i$. Next, since $d \geq 2$, we can find $\dot{q}$ solving the equations $\dot{q}(y_i) + \nabla q(y_i)^\top \dot{y}_i = \dot{s}_i$. Indeed, we have $k+1$ equations in the coefficients of $\dot{q}$ and $\dim \mathscr{P}_d = \binom{n+d}{d} \geq \binom{n+2}{2} = \frac{(n+2)(n+1)}{2} \geq k+1$. Finally, we choose $\vec{\dot{w}} = \vec{\dot{z}} - \vec{\dot{y}}$ and conclude that F_k is transverse to W_k.

By the parametric transversality theorem (Theorem D.51), for generic q, $F_k(q,-)$ is transverse to W_k. Fix such a q, let $x \in \mathbf{R}^n$, and take $k+1$ points $y_0,\dots,y_k$ from $B(x, \mathrm{dist}_Y(x)) \cap Y$. Then $F_k(q,x,\vec{y}) \in W_k$, so we can write the transversality property:

$$\mathrm{im}(DF_k(q,-)_{x,\vec{y}}) + T_{F_k(q,x,\vec{y})} W_k = T_{F_k(q,x,\vec{y})\,k+1} J^0(\mathbf{R}^n, \mathbf{R}^2),$$

or more explicitly,

$$\mathrm{im}(DF_k(q,-)_{x,\vec{y}}) + \mathbf{R}^{n(k+1)} \times 0^{k+1} \times \mathbf{R}(1,\dots,1) = \mathbf{R}^{n(k+1)} \times \mathbf{R}^{k+1} \times \mathbf{R}^{k+1}.$$

Using the expression of DF_k above, we know that

$$DF_k(q,-)_{x,\vec{y}} \colon (\dot{x}, \vec{\dot{y}}) \mapsto (\vec{\dot{y}}, \dots, \nabla q(y_i)^\top \dot{y}_i, \dots, 2(x-y_i)^\top(\dot{x}-\dot{y}_i), \dots).$$

In particular, when the terms $\nabla q(y_i)^\top \dot{y}_i$ equal 0, then we also have $(x-y_i)^\top \dot{y}_i = 0$ because $x - y_i$ and $\nabla q(y_i)$ are collinear. Thus we need $((x-y_i)^\top \dot{x})_i$ to span a supplementary space to $\mathbf{R}(1,\dots,1)$ (by a dimension argument). For this last part to be satisfied, we need the vectors

$$\{(x-y_i)-(x-y_0)\}_{i=1,\dots,k} = \{y_i - y_0\}_{i=1,\dots,k}$$

to be linearly independent. This is equivalent to $y_0,\dots,y_k$ forming a nondegenerate simplex in $\mathbf{R}^n$.

Finally, we pick q transverse to every W_k for $k \in \{1,\dots,n+1\}$: this is a generic choice. By the above arguments, any distinct set of at most $n+2$ points in $B(x, \mathrm{dist}_Y(x)) \cap Y$ forms a nondegenerate simplex. We conclude that the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ has at most $n+1$ points (any more would lead to a contradiction with nondegeneracy) and that these points form a nondegenerate simplex. $\square$

D.4.3 Finite number of critical points

In this section we study the structure of critical points for the function $\operatorname{dist}_Y|_X$ for generic $X = Z(p)$ and $Y = Z(q)$.

D.57 **Definition** (k-critical points). Let X be a subset of $\mathbf{R}^n$ and Y a closed definable subset of $\mathbf{R}^n$. For all $k \geq 0$, we define the **k-critical points of** $\operatorname{dist}_Y|_X$ as the critical points $x \in X$ of $\operatorname{dist}_Y|_X$ such that $B(x, \operatorname{dist}_Y(x)) \cap Y = \{y_0, \dots, y_k\}$. We denote by $C_{k,*}(\operatorname{dist}_Y|_X)$ the set of k-critical points of $\operatorname{dist}_Y|_X$ that are not in Y. ■

Note that, by Theorem D.56, every critical point is a k-critical point for some $k \in \{0, \dots, n\}$.

D.58 **Definition** (Algebraic critical points). Let $d \geq 0$, $p, q \in \mathscr{P}_d$, and $k \geq 0$. We define the algebraic set

$$C_k(p,q) := \left\{ \begin{array}{c|l} & x = \mu \nabla p(x) + \sum_{i=0}^k \lambda_i y_i, \\ x \in \mathbf{R}^n, & \sum_{i=1}^k \lambda_i = 1, \\ y_0, \dots, y_k \in \mathbf{R}^{n(k+1)}, & p(x) = 0, \\ \lambda_0, \dots, \lambda_k \in \mathbf{R}^{k+1}, & \forall i \in \{0, \dots, k\},\ q(y_i) = 0, \\ \mu, r \in \mathbf{R} & \forall i \in \{0, \dots, k\},\ \|x - y_i\|^2 = r^2, \\ & \forall i \in \{0, \dots, k\},\ \operatorname{rank}(x - y_i, \nabla q(y_i)) \leq 1 \end{array} \right\}.$$

We then define the **algebraic k-critical points w.r.t.** p **and** q as the set of points $x \in \mathbf{R}^n$ in the projection of $C_k(p,q)$ on the first component. ■

D.59 **Lemma.** *For generic $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$, every critical point x of* $\operatorname{dist}_Y|_X$ *not in Y is an algebraic k-critical point for some $k \in \{0, \dots, n\}$.*

Proof. Let x be an element of $X \setminus Y$. By Corollary D.20 and Proposition D.21, we have

$$\partial_x f = \operatorname{proj}_{T_x X}\left(\operatorname{co}\left\{ \frac{x - y}{\|x - y\|} \,\middle|\, y \in B(x, \operatorname{dist}_Y(x)) \cap Y \right\} \right).$$

Recall that x is a critical point if 0 belongs to this set. By Theorem D.56, for generic p and q, the set $B(x, \operatorname{dist}_Y(x)) \cap Y$ is finite with at most $n+1$ elements.

Moreover, for each point y in $B(x, \mathrm{dist}_Y(x)) \cap Y$, the function $y \mapsto \|x-y\|$ reaches a local minimum at x, so $x-y$ is parallel to $\nabla q(y)$.

Therefore, if x is a critical point of $\mathrm{dist}_Y|_X$, then there exist $k \in \{0, \dots, n\}$, $y_0, \dots, y_k$ in Y, μ in $\mathbf{R}$, and $\lambda_0, \dots, \lambda_k$ in $[0,1]$ such that

$$\begin{gathered} \sum_{i=0}^{k} \lambda_i = 1, \\ \sum_{i=0}^{k} \lambda_i (x - y_i) = \mu \nabla p(x), \\ \forall i \in \{1, \dots, k\},\ \|x - y_i\| = \|x - y_0\|, \\ \forall i \in \{0, \dots, k\},\ \mathrm{rank}(x - y_i, \nabla q(y_i)) \le 1, \end{gathered} \tag{D.11}$$

and so we conclude that $(x, \vec{y}, \vec{\lambda}, \mu, \mathrm{dist}_Y(x)) \in C_k(p,q)$ and x is an algebraic k-critical point. □

D.60 **Theorem** (Finite critical points). *For $d_1, d_2 \ge 3$ and $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$, let $X = Z(p)$ and $Y = Z(q)$. Then, for the generic choice of p and q, there are only a finite number of critical points of $\mathrm{dist}_Y|_X$. Moreover, for every $k \in \{0, \dots, n\}$, the number of k-critical points of $\mathrm{dist}_Y|_X$ not on $X \cap Y$ is bounded by*

$$\# C_{k,*}(\mathrm{dist}_Y|_X) \le c(k,n) d_1^n d_2^{n(k+1)},$$

for some $c(k,n) > 0$ depending on n and k only.

Proof. Consider the map

$$F^1 : \left\{ \begin{array}{ccc} \begin{array}{c} \mathscr{P}_{d_1} \times \mathscr{P}_{d_2} \times \mathbf{R}^n \times \\ (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L \end{array} & \to & J^1(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^L \\ (p, q, x, \vec{y}, \vec{\lambda}, \mu, r) & \mapsto & \begin{array}{c} (x, p(x), \nabla p(x), \\ \vec{y}, q(\vec{y}), \nabla q(\vec{y}), \vec{\lambda}, \mu, r), \end{array} \end{array} \right.$$

where Δ is the fat diagonal and $L = k+3$. Consider the following subspace of

$J^1(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^L$:

$$W^1 := \left\{ \begin{array}{c|l} & x = \mu u + \sum_{i=0}^{k} \lambda_i y_i, \\ (x, s, u) \in J^1(\mathbf{R}^n, \mathbf{R}), & \sum_{i=0}^{k} \lambda_i = 1, \\ (\vec{y}, \vec{t}, \vec{v}) \in {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}), & s = 0, \\ \vec{\lambda} \in \mathbf{R}^{k+1}, & \forall i \in \{0, \ldots, k\},\ t_i = 0, \\ \mu, r \in \mathbf{R} & \forall i \in \{0, \ldots, k\},\ \|x - y_i\|^2 = r^2, \\ & \forall i \in \{0, \ldots, k\},\ \operatorname{rank}(x - y_i, v_i) \le 1 \end{array} \right\}.$$

We observe that the projection of $\operatorname{im} F^1(p, q, -) \cap W^1$ onto the component x contains the set of algebraic k-critical points w.r.t. p and q. Checking the equations in the definition of W^1 from the bottom up, we see that they are independent, so W^1 has codimension

$$\begin{aligned} \operatorname{codim} W^1 &= (k+1)(n-1) + (k+1) + (k+1) + 1 + 1 + n \\ &= nk + 2n + k + 3 \\ &= \dim(\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L). \end{aligned}$$

By Theorem D.54 and Example D.55, for $d_1, d_2 \ge 3$, the map F^1 is a submersion. By Corollary D.52, for generic p, q the set of algebraic k-critical points is an algebraic set of dimension 0, so it is finite. By Lemma D.59, we conclude that the number of critical points is finite.

In order to estimate the cardinality of $C_{k,*}(\operatorname{dist}_Y|_X)$ under the assumptions of the statement, we observe that this set is the projection on the x-coordinate of the set $\Lambda_k \subset \mathbf{R}^n \times \mathbf{R}^{n(k+1)} \times \mathbf{R}^{k+1} \times \mathbf{R}$ consisting of the points $(x, \vec{y}, \vec{\lambda}, \mu)$ solving the system (D.11) with $x \in Z(p)$, $y_i \in Z(q)$ for all $i \in \{0, \ldots, k\}$. Since $C_{k,*}(\operatorname{dist}_Y|_X)$ is finite, its cardinality equals the number of its connected components; moreover each component of Λ_k projects to a component of $C_{k,*}(\operatorname{dist}_Y|_X)$ and we have

$$\# C_{k,*}(\operatorname{dist}_Y|_X) = b_0(C_{k,*}(\operatorname{dist}_Y|_X)) \le b_0(\Lambda_k) \le b(\Lambda_k),$$

where $b(\Lambda_k)$ denotes the sum of the Betti numbers of Λ_k. In order to control this we use [4, Theorem 2.6] which states that the sum of the Betti numbers of an algebraic set $V \subset \mathbf{R}^{n_1 + \cdots + n_a}$ defined by ℓ polynomials $p_j(x_1, \ldots, x_a)$ of degree d_i in the variable $x_i \in \mathbf{R}^{n_i}$ can be bounded by

$$b(V) \le O(1)^{n_1 + \cdots + n_a} a^{3(n_1 + \cdots + n_a)} d_1^{n_1} \cdots d_p^{n_a}.$$

In our case $a=4$, $\vec{n}=(n,n(k+1),k+1,1)$ and $\vec{d}=(\deg(p),\deg(q),1,1)$, so that

$$b(\Lambda_k)\le c(k,n)\deg(p)^n\deg(q)^{n(k+1)}. \qquad \square$$

D.61 **Remark.** The same proof works also in the case $(X,Y)=(\mathbf{R}^n,Z(q))$ (see also Remark D.70), in which case (following the above notation) $\vec{d}=(1,\deg(q),1,1)$. The statement in this case is the following. For the generic polynomial $q\in\mathscr{P}_d$ the function $\mathrm{dist}_{Z(q)}$ has only finitely many critical points, all these critical points are algebraic and the number of algebraic k-critical points is bounded by

$$\#C_{k,*}(\mathrm{dist}_{Z(q)})\le\tilde{c}(k,n)d^{n(k+1)}.$$

In particular this recovers and generalizes to all dimensions the bound $\#C_{1,*}\le c(n)d^{2n}$ that one can obtain by combining [27, Corollary 4.5] with the EDD estimate [24, Corollary 2.10]. ■

D.4.4 Continuous selections

D.62 **Proposition.** *Let $x\in\mathbf{R}^n$. For $d_1\ge 3$ and $d_2\ge 4$ and generic $p\in\mathscr{P}_{d_1}$ and $q\in\mathscr{P}_{d_2}$, the critical points of $\mathrm{dist}_Y|_X$ are regular values of the normal exponential map of Y.*

Proof. The normal exponential map of Y is

$$\exp\colon\begin{cases} NY\cong Y\times\mathbf{R} & \to \quad \mathbf{R}^n\\ (y,t) & \mapsto \quad y+t\nabla q(y).\end{cases}$$

Its differential at a point $(y,t)\in NY$ is

$$D\exp_{(y,t)}\colon\begin{cases} TNY\subseteq\mathbf{R}^n\times\mathbf{R} & \to \quad T\mathbf{R}^n\cong\mathbf{R}^n\\ (\dot{y},\dot{t}) & \mapsto \quad \dot{y}+tH(q)(y)\dot{y}+\dot{t}\nabla q(y).\end{cases}$$

Thus a point x in $\mathbf{R}^n$ is a critical value of exp if, and only if, it is equal to $y+t\nabla q(y)$ for some (y,t) in NY such that the block matrix $[\mathrm{Id}+tH(q)(y),\nabla q(y)]$ restricted to $\nabla q(y)^\perp\times\mathbf{R}$ has rank strictly less than n. Consider the map

$$F^2\colon\begin{cases} \begin{matrix}\mathscr{P}_{d_1}\times\mathscr{P}_{d_2}\times\mathbf{R}^n\times\\ (\mathbf{R}^{n(k+1)}\smallsetminus\Delta)\times\mathbf{R}^L\end{matrix} & \to \quad J^1(\mathbf{R}^n,\mathbf{R})\times{}_{k+1}J^2(\mathbf{R}^n,\mathbf{R})\times\mathbf{R}^L\\ (p,q,x,\vec{y},\vec{\lambda},\mu,r) & \mapsto \quad \begin{matrix}(x,p(x),\nabla p(x),\\ \vec{y},q(\vec{y}),\nabla q(\vec{y}),H(q)(\vec{y}),\vec{\lambda},\mu,r),\end{matrix}\end{cases}$$

and let

$$W^2 := \left\{ \begin{array}{l} (x,s,u,\vec{y},\vec{t},\vec{v},\vec{H},\vec{\lambda},\mu,r) \in J^1(\mathbf{R}^n,\mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}) \times \mathbf{R}^L : \\ (x,s,u,\vec{y},\vec{t},\vec{v},\vec{\lambda},\mu,r) \in W^1 \end{array} \right\},$$

where W^1 is defined in the proof of Theorem D.60. We define the subspace of $J^1(\mathbf{R}^n,\mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}) \times \mathbf{R}^L$

$$W^2_{\text{crit}} := \left\{ \begin{array}{l} (x,s,u,\vec{y},\vec{t},\vec{v},\vec{H},\vec{\lambda},\mu,r) \in W^2 : \\ \forall i \in \{0,\dots,k\},\ \operatorname{rank}[(\operatorname{Id}+rH_i)(\operatorname{Id}-v_iv_i^\top),v_i] < n \end{array} \right\}.$$

This space is defined by the same equations as W^1 and additional equations that involve the H_i. These new equations are independent of those defining W^2. Indeed, given $(x,s,u,\vec{y},\vec{t},\vec{v},\vec{H},\vec{\lambda},\mu,r)$ in W^2 and $i \in \{0,\dots,k\}$, we can assume $H_i = 0$, which gives us $\operatorname{rank}[(\operatorname{Id}-v_iv_i^\top),v_i] = n$.

We deduce that W^2_{crit} has codimension $> nk+2n+k+3$, which is the dimension of $\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L$. By Theorem D.54 and Example D.55, for $d_1 \geq 3$ and $d_2 \geq 4$, the map F^2 is a submersion. By Corollary D.52, for generic p and q, the critical points of $\operatorname{dist}_Y|_X$ miss W^2_{crit}. We conclude that, for generic p and q, the critical points of $\operatorname{dist}_Y|_X$ are regular values for the normal exponential map of Y. □

D.63 **Corollary.** *For generic $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$ of respective degrees $d_1 \geq 3$ and $d_2 \geq 4$, denoting by $X = Z(p)$ and $Y = Z(q)$ the function $\operatorname{dist}_Y|_X$ is a continuous selection of $\mathscr{C}^2$ functions around its critical points.*

Proof. By Proposition D.62, for generic p and q, the critical points of $\operatorname{dist}_Y|_X$ are regular values of the normal exponential map of Y. By Proposition D.36, we deduce that $\operatorname{dist}_Y|_X$ is a $\mathscr{C}^2$ selection around its critical points. □

D.4.5 Nondegeneracy of critical points

Now we aim to describe the degeneracy of critical points algebraically. To do this, we need to describe the Hessian of $\operatorname{dist}_Y|_X$ at x in terms of the derivatives of p and q.

D.64 **Lemma.** *Let $Y = Z(q)$ and $y_0 \in Y$. Then the second fundamental form of Y at y_0, is*

$$\mathrm{II}_{y_0} : \left\{ \begin{array}{ccc} T_{y_0}Y \times T_{y_0}Y & \to & N_{y_0}Y = \mathbf{R}\nabla q(y_0) \\ (v,w) & \mapsto & -\frac{\nabla q(y_0)}{\|\nabla q(y_0)\|} v^\top H(q)(y_0) w. \end{array} \right.$$

Proof. Let $v_0, w_0 \in T_{y_0}Y$ be tangent vectors and write $\alpha := \|\nabla q(y_0)\|$. Define the tangent vector field

$$v\colon y \mapsto \frac{1}{\alpha^2}\Big(v_0\|\nabla q(y)\|^2 - (v_0^\top \nabla q(y))\nabla q(y)\Big)$$

in a neighborhood around y_0. Note that $v(y) \in T_yY$ for all y and $v(y_0) = v_0$. By one definition of the second fundamental form [40, Chapter 8], we have

$$\mathrm{II}_y(v_0, w_0) = \frac{\nabla q(y_0)}{\alpha}\nabla q(y_0)^\top J(v)(y_0)w_0$$

(i.e. the projection onto N_yY of the differential of v in the direction w_0), where $J(v)$ is the Jacobian of v.

Recall that $J(v)$ is the matrix $(\frac{\partial v_i}{\partial x_j})_{1\le i,j\le n}$, where v_i is the i^{th} component of $v(y)$. For all $i \in \{1,\dots,k\}$, we have

$$\alpha^2 v_i(y) = v_{0,i}\|\nabla q(y)\|^2 - (v_0^\top \nabla q(y))\frac{\partial q}{\partial x_i}(y).$$

Then, for all $j \in \{1,\dots,n\}$, we have

$$\begin{aligned}\alpha^2\frac{\partial v_i}{\partial x_j}(y) = v_{0,i}\cdot 2\nabla q(y)^\top \frac{\partial \nabla q}{\partial x_j}(y) - \left(v_0^\top \frac{\partial \nabla q}{\partial x_j}(y)\right)\frac{\partial q}{\partial x_i}(y)\\ - (v_0^\top \nabla q(y))\frac{\partial^2 q}{\partial x_i x_j}(y).\end{aligned}$$

Thus we can write

$$\alpha^2\nabla v_i(y)^\top = 2v_{0,i}\nabla q(y)^\top H(q)(y) - (v_0^\top H(q)(y))\frac{\partial q}{\partial x_i}(y) - (v_0^\top \nabla q(y))\frac{\partial \nabla q}{\partial x_i}(y)^\top$$

and

$$\alpha^2 J(v)(y) = 2v_0\nabla q(y)^\top H(q)(y) - \nabla q(y)v_0^\top H(q)(y) - (v_0^\top \nabla q(y))H(q)(y).$$

Finally, we get

$$\begin{aligned}\mathrm{II}_{y_0}(v_0, w_0) &= \frac{\nabla q(y_0)}{\alpha}\nabla q(y_0)^\top J(v)(y_0)w_0\\ &= \frac{\nabla q(y_0)}{\alpha^3}\Big(2(\nabla q(y_0)^\top v_0)\nabla q(y_0)^\top - \nabla q(y_0)^\top \nabla q(y_0)v_0^\top\\ &\quad - \nabla q(y_0)^\top(v_0^\top \nabla q(y_0))\Big)H(q)(y_0)w_0\\ &= -\frac{\nabla q(y_0)}{\alpha}v_0^\top H(q)(y_0)w_0,\end{aligned}$$

using the fact that $v_0^\top \nabla q(y_0) = 0$. ☐

For all y in Y, we define $g_y := \operatorname{dist}_{B(y,\varepsilon)\cap Y}$ and $f_y := g_y|_X$ for $y \in Y$. We also introduce the following notation: for a symmetric matrix $H \in \operatorname{Sym}(n, \mathbf{R})$ and a subspace $V \subseteq \mathbf{R}^n$, we denote by $H|_V$ a matrix representing the restriction of the quadratic form $h(x) = x^\top H x$ to V. If $L = [v_1, \ldots, v_\ell] \in \mathbf{R}^{n\times \ell}$ is a matrix whose columns form a basis for V, then one can take

$$H|_V = L^\top H L.$$

D.65 **Definition** (Algebraic degenerate k-critical points). The set of **algebraic degenerate k-critical points w.r.t.** p **and** q is the projection of the set

$$D_k(p,q) := \left\{ \begin{array}{l} (x, \vec{y}, \vec{\lambda}, \mu, r) \in C_k(p,q): \\ \det\left(\mu H(p)(x)|_{\partial_x f^\perp} + r\sum_{i=0}^k \lambda_i H(g_{y_i})(x)|_{\partial_x f^\perp}\right) = 0 \end{array} \right\} \tag{D.12}$$

onto the first component x. ■

The use of the adjective “algebraic” is justified by the following lemma.

D.66 **Lemma.** *For all $k \in \{1, \ldots, n\}$, the set $D_k(p,q)$ is algebraic.*

Proof. Let $(x, \vec{y}, \vec{\lambda}, r) \in D_k(p,q)$. For all $i \in \{0, \ldots, k\}$, write $\nu_i := \frac{\nabla q(y_i)}{\|\nabla q(y_i)\|}$ the unit normal vector from Y at y_i. By Lemma D.64, we have the second fundamental form II_{y_i} of Y at y_i, defined as a bilinear form from $T_{y_i}Y$ to $N_{y_i}Y$. Taking the scalar product with ν_i gives a quadratic form represented by the matrix $-H(q)(y_i)/\|\nabla q(y_i)\|$. We extend the matrix representation of this form to $N_{y_i}Y$ as follows:

$$Q_i := (\mathrm{Id} - \nu_i\nu_i^\top)\frac{-H(q)(y_i)}{\|\nabla q(y_i)\|}(\mathrm{Id} - \nu_i\nu_i^\top),$$

with Id the identity matrix (note that the orientation of ν_i does not matter). By [33, Lemma 14.17], for each $i \in \{0, \ldots, k\}$, the Hessian of g_{y_i} and Q_i are diagonalizable in the same basis (one consisting of the directions of principal curvature of Y and of a normal vector, see also Section D.3.1), and the eigenvalues of $H(g_{y_i})$ are the image of those of Q_i via the function $\kappa \mapsto \frac{-\kappa}{1-\|x-y_i\|\kappa}$. We deduce that

$$H(g_{y_i})(x) = \frac{-Q_i}{1 - \|x - y_i\| Q_i}.$$

Let $\nu := \frac{\nabla p(x)}{\|\nabla p(x)\|}$ and $V := [\nu, \nu_0, \dots, \nu_k] \in \mathbf{R}^{n\times(k+2)}$ be the matrix of normal vectors. We want to project $H(p)(x)$ and the $H(g_{y_i})(x)$ onto $\partial_x f^\perp \cap \nabla p(x)^\perp$ and then take the convex combination. Since we end up with an $n \times n$ matrix, the equation in (D.12) becomes

$$\operatorname{rank}\left(\mu L^\top H(p)(x)L + r\sum_{i=0}^{k}\lambda_i L^\top H(g_{y_i})(x)L\right) \leq n-k-3,$$

where $L = \mathrm{Id} - V(V^\top V)^{-1}V^\top$, and this equation is algebraic. □

D.67 **Lemma.** *For generic $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$ with $d_1 \geq 3$ and $d_2 \geq 4$, every degenerate critical point of* $\mathrm{dist}_Y|_X$ *not in Y is an algebraic degenerate k-critical point for some $k \in \{1, \dots, n\}$.*

Proof. We pick generic p and q such that, for all x in X, the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is a nondegenerate simplex (Theorem D.56) and that $\mathrm{dist}_Y|_X$ is a $\mathscr{C}^2$ selection around each of its critical points (Corollary D.63). In this context, a critical point is degenerate if, and only if, condition (2) of Definition D.31 holds.

Explicitly, let x be a critical point of $\mathrm{dist}_Y|_X$ not in Y and write $\{y_0, \dots, y_k\} = B(x, \mathrm{dist}_Y(x)) \cap Y$. For all $i \in \{0, \dots, k\}$, the differential of f_{y_i} at x is the projection of $\nabla g_{y_i}(x)$ on T_xX: explicitly, there exists $\mu_i \in \mathbf{R}$ such that

$$\begin{aligned}\nabla f_{y_i}(x) &= \nabla g_{y_i}(x) - \mu_i \nabla p(x) && \text{(D.13)}\\ &= \frac{x-y_i}{\|x-y_i\|} - \mu_i \nabla p(x). && \text{(D.14)}\end{aligned}$$

Since, by definition of critical points, there exist λ_i such that $\sum \lambda_i = 1$ and $\sum \lambda_i \nabla f_{y_i}(x) = 0$, we get

$$x = \sum_{i=0}^{k}(\lambda_i y_i + \lambda_i \mu_i \|x - y_i\| \nabla p(x)).$$

Let $\mu := \sum_I \lambda_i \mu_i \|x - y_i\| = \mathrm{dist}_Y(x) \sum_i \lambda_i \mu_i$. We deduce that $(x, \vec{y}, \vec{\lambda}, \mu, \mathrm{dist}_Y(x))$ is an element of $C_k(p,q)$.

We can write $\mathrm{dist}_Y|_X$ as the continuous selection of $\mathscr{C}^2$ functions

$$\mathrm{dist}_Y|_X = \min_{i=0,\dots,k} f_{y_i}.$$

Therefore, using Remark D.32, we see that the critical point x is degenerate if, and only if, the matrix

$$\begin{aligned}
&\sum_{i=0}^{k}(\lambda_i H(g_{y_i})(x)|_{\partial_x f^\perp} + \lambda_i\mu_i H(p)(x)|_{\partial_x f^\perp}) \\
&= \frac{\mu}{\mathrm{dist}_Y(x)} H(p)(x)|_{\partial_x f^\perp} + \sum_{i=0}^{k}\lambda_i H(g_{y_i})(x)|_{\partial_x f^\perp},
\end{aligned}$$

is singular. So we conclude that if x is a degenerate critical point of $\mathrm{dist}_Y|_X$, then $(x, \vec{y}, \vec{\lambda}, \mu, \mathrm{dist}_Y(x))$ is an element of $D_k(p,q)$, and so x is an algebraic degenerate k-critical point for $k \leq n$. □

D.68 **Remark.** The Morse index of a nondegenerate critical point of $\mathrm{dist}_Y|_X$ in the case $X = Z(p)$ and $Y = \{y\}$ with $y \notin X$ can be computed using the Lagrange's multipliers rule, as follows. First, a critical point of $\mathrm{dist}_Y|_X$ is a point $x \in X$ where

$$\lambda \nabla p(x) + y - x = 0$$

for some $\lambda \neq 0$. Now, assuming that x is a nondegenerate critical point, the matrix representing the Hessian of $\mathrm{dist}_Y|_X$ at x in the basis $\{v_1, \dots, v_{n-1}\}$ for $T_x X$ can be written as

$$L^\top \left(\lambda H(p)(x) - \mathrm{Id}\right) L,$$

where $L = [v_1, \dots, v_{n-1}]$. ■

D.69 **Theorem** (Genericity of nondegeneracy). *For generic $p \in \mathscr{P}_{d_1}$ and $q \in \mathscr{P}_{d_2}$ with $d_1, d_2 \geq 4$, all critical points of* $\mathrm{dist}_Y|_X$ *not in Y are nondegenerate.*

Proof. Consider the map

$$F^3 : \begin{cases} \begin{array}{c} \mathscr{P}_{d_1} \times \mathscr{P}_{d_2} \times \mathbf{R}^n \times \\ (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L \end{array} & \longrightarrow & J^2(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^L \\[2ex] (p, q, x, \vec{y}, \vec{\lambda}, \mu, r) & \longmapsto & \begin{array}{c} (x, p(x), \nabla p(x), H(p)(x), \\ \vec{y}, q(\vec{y}), \nabla q(\vec{y}), H(q)(\vec{y}), \vec{\lambda}, \mu, r), \end{array} \end{cases}$$

and let

$$W^3 := \left\{ \begin{array}{l} (x, s, u, G, \vec{y}, \vec{t}, \vec{v}, \vec{H}, \vec{\lambda}, \mu, r) \in J^2(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^L : \\ (x, s, u, \vec{y}, \vec{t}, \vec{v}, \vec{\lambda}, \mu, r) \in W^1 \end{array} \right\},$$

where W^1 is defined in the proof of Theorem D.60.

Let us find a system equations of equations such that, for generic p and q, elements of $F^3(\{p\} \times \{q\} \times D_k(p,q))$ satisfy these equations. We define the subspace of $J^2(\mathbf{R}^n,\mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}) \times \mathbf{R}^L$

$$W^3_{\deg} := \left\{(x,s,u,G,\vec{y},\vec{t},\vec{v},\vec{H},\vec{\lambda},\mu,r) \in W^3 \;\middle|\; \begin{array}{l} \forall i \in \{0,\dots,k\},\ \nu_i := \frac{v_i}{\|v_i\|} \in \mathbf{R}^n, \\ \forall i \in \{0,\dots,k\},\ Q_i := (\mathrm{Id} - \nu_i\nu_i^\top)\frac{H_i}{\|v_i\|}(\mathrm{Id} - \nu_i\nu_i^\top) \in \mathbf{R}^{n\times n}, \\ \forall i \in \{0,\dots,k\},\ \tilde{H}_i := \frac{-Q_i}{1-rQ_i} \in \mathbf{R}^{n\times n}, \\ V := [\frac{u}{\|u\|}, v_0, \dots, v_k] \in \mathbf{R}^{n\times(k+2)}, \\ L := \mathrm{Id} - V(V^\top V)^{-1}V^\top \in \mathbf{R}^{n\times n}, \\ \mathrm{rank}\left(\mu L^\top G L + r\sum_{i=0}^k \lambda_i L^\top \tilde{H}_i L\right) \le n-k-3 \end{array}\right\}.$$

We need to check that the last inequality is independent of the other equations defining W^3. It suffices to find an element of W^3 such that this rank is $n-k-2$. Since the equations defining W^3 do not involve G nor $\vec{H}$, we can choose these terms freely. Let $(x,s,u,\vec{y},\vec{t},\vec{v},\vec{H},\vec{\lambda},\mu,r)$ be an element of W^3. Without loss of generality, we assume $\lambda_0 \neq 0$ and we set $G = H_1 = \cdots = H_n = 0$. Up to scaling, we can assume that $r \neq 1$, and up to orthogonal transformation of H_0, we can assume ν_0 is e_n, the last element of the canonical basis of $\mathbf{R}^n$. In this case, $\|v_0\| = 1$, so $Q_0 = \mathrm{diag}(1,\dots,1,0)$ and $\tilde{H}_0 = \frac{-1}{1-r}Q_0$. Note that $\tilde{H}_0^\perp = \mathbf{R}e_n = \mathbf{R}\nu_0$, so the rank of $\lambda_0 L^\top \tilde{H}_0 L$ is maximal; that is, its rank is $n-k-2$.

We deduce that $W^3_{\deg}$ has codimension $> nk+2n+k+3$, which is the dimension of $\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L$. By Theorem D.54 and Example D.55, for $d_1, d_2 \ge 4$, the map F^3 is a submersion. By Corollary D.52, for generic p and q, the image $\mathrm{im}\, F^3(p,q,-)$ misses $W^3_{\deg}$. In particular, the image of $D_k(p,q)$ by $F^3(p,q,-)$ misses $W^3_{\deg}$. By Lemmas D.66 and D.67, we conclude that, for generic p and q, the critical points of $\mathrm{dist}_Y|_X$ not in Y are nondegenerate. □

D.70 **Remark.** The proofs throughout Section D.4 do not work as-is for the case $X = \mathbf{R}^n$ because we require p to be of degree at least 3. However, the results are still true, as can be verified by running through the proofs of Theorems D.60 and D.69 and Proposition D.62 with the parameters p, μ, s, u, and G removed from the definition of W^1, W^2, and W^3, and noting that Theorem D.54 can still be applied. ■

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Paper E

Nondegeneracy of distance functions between complete intersections

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Nondegeneracy of distance functions between complete intersections

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Abstract

We extend the results of Guidolin, Lerario, Ren, and Scolamiero (arXiv:2402.08639) to show that the distance function between generic real complete intersections, viewed as a continuous selection of locally Lipschitz functions, is nondegenerate in a Morse-theoretic sense. We also give an upper bound for the number of critical points of such distance functions.

E.1 Introduction

E.1.1 Distance functions and critical points

This paper extends the results of [11] to distance functions

$$\mathrm{dist}_Y|_X \colon X \to \mathbf{R}$$

between two real algebraic varieties X and Y in $\mathbf{R}^n$. The original paper considers distance functions as locally Lipschitz continuous selections of differentiable functions [13], defines critical points following [7], and studies the Morse theory for these functions presented in [1]. We direct the reader to the introduction of [11] for a more detailed history of the study of distance functions.

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In this paper, we focus on the theory of distance functions between algebraic varieties. These functions, and in particular their critical points, have been studied previously in specific cases. The Euclidean distance degree, introduced in [9], is the number of complex critical points of the distance function from a generic point in ambient space, restricted to a real algebraic variety. The bottleneck degree, introduced in [8], is the number of complex critical points of the distance function on a real algebraic variety, and can be seen as the number of complex lines orthogonal to two or more points of the variety. Our results generalize these two notions in the real setting.

Outside of the algebraic context, we would like to highlight the following works. [16], an earlier study on properties of critical points of distance functions from generically embedded manifolds, was the basis for Theorem E.17. [6] studies the distance function from a Riemannian manifold to a single point, while [4] considers the distance function from $\mathbf{R}^n$ to a finite set of points. More recently, [15] defines a Morse theory (slightly different from ours) for distance functions from a smooth bounded surface, and [2] show that the number of critical points of the distance function from a generically embedded manifold is finite.

E.1.2 Main results

In order to develop a Morse theory for distance functions, we first need a notion of differential. This is provided by Clarke's **subdifferential** (Definition E.1), which we define more generally for locally Lipschitz functions as the convex hull of all limits of gradients when converging to a point. This allows us to define **critical points** as points where the subdifferential contains 0 (Definition E.3). In order to talk about nondegenerate critical points, we restrict our functions to **continuous selections** of $\mathscr{C}^2$ functions (Definition E.4): a continuous selection from a finite set $\{f_0,\dots,f_m\} \subseteq \mathscr{C}^2(X,\mathbf{R})$ functions is a continuous function $f\colon X \to \mathbf{R}$ that is equal to one (or more) of the f_i at each point $x \in X$. With this, we can finally define **nondegenerate** critical points for continuous selections (Definition E.5).

Given these definitions, we show that, generically, the distance function $\operatorname{dist}_Y|_X$ between two complete intersections has no degenerate critical points, and that it has a finite number of (nondegenerate) critical points. To properly state this, we first need to show that $\operatorname{dist}_Y|_X$ is a continuous selection. Moreover, to make sense of the genericity claim, we consider complete intersections: for all $m \in \{1,\dots,n\}$ and $d \geq 0$, we denote by $\mathscr{P}_d$ the space of real polynomials of degree at most d and $\mathscr{C}_d^m$ the open subset of $(\mathscr{P}_d)^m$ whose elements generate complete intersections (see Definition E.16). As in [11], we collect our main results into one theorem:

Theorem. *Fix $\ell, m \in \{1,\dots,n\}$ and $d_1, d_2 \geq 0$. Let $\vec{p} \in \mathscr{C}_{d_1}^\ell$ and $\vec{q} \in \mathscr{C}_{d_2}^m$, and let $X = V(\vec{p})$ and $Y = V(\vec{q})$ be the defined varieties. Then, generically,*

1. *if $d_2 \geq 2$, then for every $z \in \mathbf{R}^n$ the set $B(z, \operatorname{dist}_Y(z)) \cap Y$ of closest points in Y to z forms a nondegenerate simplex: in particular, there are at most $n+1$ closest points (Theorem E.17);*
2. *if $d_1, d_2 \geq 3$, then $\operatorname{dist}_Y|_X$ has a finite number of critical points (Theorem E.21); in particular, for all $k \in \{1,\dots,n\}$, denoting by $C_{k,*}(\operatorname{dist}_Y|_X)$ the set of critical points in $X \setminus Y$ with exactly $k+1$ closest points (Definition E.18), we have*
$$\#C_{k,*}(\operatorname{dist}_Y|_X) \leq c(k,\ell,m,n) d_1^n d_2^{n(k+1)},$$
for some constant $c(k,\ell,m,n) > 0$ depending on k, ℓ, m, and n only;
3. *if $d_1 \geq 3$ and $d_2 \geq 4$, then the function $\operatorname{dist}_Y|_X$ is a continuous selection near each of its critical points (Corollary E.24);*
4. *if $d_1, d_2 \geq 4$, then every critical point of $\operatorname{dist}_Y|_X$ is nondegenerate (Theorem E.29).*

Analogous results hold when $X = \mathbf{R}^n$: see Remarks E.22 and E.30. In summary, the distance function between complete intersections is generically nondegenerate (i.e., its critical points are all nondegenerate) in a Morse-theoretic sense.

E.1.3 Structure of the paper

In Section E.2 we recall useful results from [11]. In Section E.3 we prove the main results, closely following the structure of [11, Section 4]. This paper, in particular Section E.3, is quite notation-heavy; Appendix E.A provides a broad overview of the notation conventions used throughout the text.

E.1.4 Acknowledgments

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E.2 Preliminaries

E.2.1 Critical points for locally Lipschitz functions

In this section we recall the notions that will allow us to study distance functions in a Morse-theoretic way: subdifferentials, critical points, and nondegeneracy. The definitions and results of this section come from [11].

E.1 **Definition** (Subdifferential). Let X be a Riemannian manifold and $f\colon X \to \mathbf{R}$ a locally Lipschitz function. The **subdifferential of** f **at** $x \in X$, denoted by $\partial_x f$, is the convex body defined by

$$\partial_x f := \operatorname{conv}\left\{ \lim_{x_k \to x, x_k \in \Omega(f)} \nabla f(x_k) \;\middle|\; \text{the limit exists} \right\} \subseteq T_x X.$$

Note that at a strictly differentiable point, the subdifferential reduces to a singleton $\partial_x f = \{\nabla f(x)\}$. ■

E.2 **Proposition** ([11]). *Let $Y \subseteq \mathbf{R}^n$ be closed and $X \subseteq \mathbf{R}^n$ a smooth manifold. Then, for every $x \in X \setminus Y$,*

$$\partial_x (\operatorname{dist}_Y|_X) = \operatorname{proj}_{T_x X}\left(\operatorname{conv}\left\{ \frac{x-y}{\|x-y\|} \;\middle|\; y \in B(x, \operatorname{dist}_Y(x)) \cap Y \right\} \right),$$

where $\operatorname{proj}_{T_x X}\colon \mathbf{R}^n \to T_x X$ *denotes the orthogonal projection.*

E.3 **Definition** (Critical points). We say that $x \in X$ is a **critical point** of a locally Lipschitz function $f\colon X \to \mathbf{R}$ if 0 belongs to $\partial_x f$. We denote by $C(f) \subseteq X$ the set of critical points of f.

A point $x \in X$ is said to be a **regular point** of f if it is not critical; a **critical value** of f is a real number $c \in \mathbf{R}$ such that $f^{-1}\{c\} \cap C(f) \neq \varnothing$ and a **regular value** of f is a real number $c \in \mathbf{R}$ which is not critical. ■

E.4 **Definition** (Continuous selections). Let X be a smooth manifold and $\{f_0, \dots, f_m\}$ a finite subset of $\mathscr{C}^2(X, \mathbf{R})$. We will say that $f\colon X \to \mathbf{R}$ is a **continuous selection** from $\{f_0, \dots, f_m\}$ if f is continuous and for every $x \in X$ there exists $i \in \{0, \dots, m\}$ such that $f(x) = f_i(x)$. In this case we will write $f \in \operatorname{CS}(f_0, \dots, f_m)$. ■

If $f \in \operatorname{CS}(f_0, \dots, f_m)$, for $i \in \{0, \dots, m\}$ we denote by $S_i \subseteq X$ the set

$$S_i := \{x \in X \mid f(x) = f_i(x)\}.$$

Then, given $x \in X$, the **effective index set** of $f \in \mathrm{CS}(f_0, \dots, f_m)$ at $x \in X$ is defined to be

$$I(x) := \left\{ i \in \{0, \dots, m\} \mid x \in \overline{\mathrm{int}(S_i)} \right\}.$$

Note that continuous selections (of continuously differentiable functions) are locally Lipschitz [13], so we can talk about their critical points using the definitions above. Again, below we assume that X is endowed with some Riemannian structure, so as to identify its tangent and cotangent bundles.

E.5 **Definition** (Nondegenerate critical points of continuous selections). Let $f \in \mathrm{CS}(f_0, \dots, f_m)$. A critical point $x \in C(f)$ is said to be **nondegenerate** (with respect to $f_0, \dots, f_m$) if the following conditions are satisfied:

1. For every $i \in I(x)$ the set of differentials $\{D_x f_j \mid j \in I(x) \smallsetminus \{i\}\}$ is linearly independent.
2. Denote by $(\lambda_i)_{i \in I(x)}$ the unique list of nonnegative real numbers such that

$$\sum_{i \in I(x)} \lambda_i = 1 \quad \text{and} \quad \sum_{i \in I(x)} \lambda_i D_x f_i = 0,$$

and by $\lambda f \colon X \to \mathbf{R}$ the function defined by $\lambda f(z) := \sum_{i \in I(x)} \lambda_i f_i(z)$. Then the second differential of λf is nondegenerate on the subspace

$$T(x) := \bigcap_{i \in I(x)} \ker(D_x f_i).$$

We denote by $k(x) := \#I(x) - 1$ and call this number the **piecewise linear index**, and by $\iota(x)$ the negative inertia index of the restriction of the second differential of λf to $T(x)$. We call $\iota(x)$ the **quadratic index**.

Clearly, in the case X is endowed with a Riemannian metric, the above conditions can be rephrased using the gradients of the f_i instead of their differential. Moreover, the nondegeneracy of the second differential of λf on $T(x)$ is equivalent to the nondegeneracy of the restriction of the corresponding quadratic form on it. ■

E.6 **Remark.** An important remark concerns the reformulation of these conditions in the case X is a submanifold of another manifold N and we are considering the restriction of a continuous selection $f = g|_X$ where $g \in \mathrm{CS}(g_0, \dots, g_k)$ with $g_i \in \mathscr{C}^2(N, \mathbf{R})$, so that

$$f_i = g_i|_X, \quad i = 0, \dots, k.$$

Suppose that X is given as the zero set of a set of transversally intersecting regular equations $p_j = 0$, with $p_j\colon N \to \mathbf{R}$ a smooth function (in Section E.3 we are interested in the case $N = \mathbf{R}^n$ and the p_j are polynomials). Then, for all $i \in \{0, \dots, k\}$,

$$D_x f_i = D_x g_i|_{T_x X} = D_x g_i|_{\bigcap_j \ker(D_x p_j)}.$$

Rewriting the above conditions with this notation, we see that $x \in X$ is a nondegenerate critical point of $f = g|_X$ if and only if the following conditions are satisfied:

1. For every $i_0 \in I(x)$ the set of differentials $\{D_x g_i|_{\bigcap_j \ker(D_x p_j)} \mid i \in I(x) \smallsetminus \{i_0\}\}$ is linearly independent;
2. Denote by $(\lambda_i)_{i \in I(x)}$ and $(\mu_j)_j$ the unique list of numbers such that the λ_i are nonnegative,

$$\sum_{i \in I(x)} \lambda_i = 1, \quad \text{and} \quad \sum_{j=1}^{\ell} \mu_j D_x p_j + \sum_{i \in I(x)} \lambda_i D_x g_i = 0, \tag{E.1}$$

   and by $\lambda f\colon X \to \mathbf{R}$ the function defined by $\lambda f(z) := \sum_{i \in I(x)} \lambda_i f_i(z)$. Then the second differential of λf is nondegenerate on the subspace

$$T(x) := \bigcap_{i \in I(x)} \ker(D_x f_i|_{\bigcap_j \ker(D_x p_j)}) = \left(\bigcap_{j=1}^{\ell} \ker(D_x p_j)\right) \cap \left(\bigcap_{i \in I(x)} \ker(D_x f_i)\right).$$

In order to compute the second differential of λf on $T(x)$ using the original functions, we proceed as follows. We consider the function

$$\widetilde{\lambda g} := \sum_{j=1}^{\ell} \mu_j p_j + \sum_{i \in I(x)} \lambda_i g_i \colon N \to \mathbf{R}.$$

Observe that, since $X = \{p_j = 0, j = 1, \dots, \ell\}$, then $\widetilde{\lambda g}|_X = \lambda f$. Moreover, by the second equation of (E.1), $\widetilde{\lambda g}$ has a critical point at $x \in N$ and therefore the second differential of its restriction to X equals the restriction of its second differential to $T_x X$. This means that the condition that the second differential of λf is nondegenerate on $T(x)$ is equivalent to the condition that

$$\sum_{j=1}^{\ell} \mu_j D_x^2 p_j + \sum_{i \in I(x)} \lambda_i D_x^2 g_i$$

is nondegenerate on $T(x)$. ∎

Denote by $\varphi\colon NY \to \mathbf{R}^n$ the restriction of the exponential map of $\mathbf{R}^n$ to the normal bundle of Y. Note that we can think of elements in NY as pairs $(y,v) \in \mathbf{R}^{2n}$ with $y \in Y$ and $v \in N_y Y$ and $\varphi(y,v) = y + v$.

E.7 **Proposition** ([11]). *Suppose that Y is smooth and let $x \in \mathbf{R}^n$ be a regular value of $\varphi\colon NY \to \mathbf{R}^n$. Then*

1. *$B(x, \mathrm{dist}_Y(x)) \cap Y$ is a finite set $\{y_0, \dots, y_k\}$,*

and for all $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

2. *for every $i \in \{0, \dots, k\}$, the function $f_i := \mathrm{dist}_{B(y_i,\varepsilon)\cap Y}$ is $\mathscr{C}^2$ on $B(x,\delta)$;*
3. *the function $\mathrm{dist}_Y|_{B(x,\delta)}$ is a continuous selection of the f_i: specifically, $\mathrm{dist}_Y|_{B(x,\delta)} = \min_i f_i$.*

E.8 **Definition.** Let $Y \subseteq \mathbf{R}^n$ be closed and definable and X be a smooth manifold. We say that a point x of X is a **nondegenerate critical point of** $\mathrm{dist}_Y|_X$ if $\mathrm{dist}_Y|_X$ is a continuous selection at x and x is a nondegenerate critical point for the continuous selection. For every $k, \iota \in \mathbf{N}$ we denote by $C_{k,\iota}(X,Y)$ the set of nondegenerate critical points of $\mathrm{dist}_Y|_X$ with piecewise linear index k and quadratic index ι and which are not in Y. ■

E.2.2 Multijet parametric transversality

In this section we prove a useful technical tool, Corollary E.13, which is the only new statement compared to [11, Section 4.1], and is only added for convenience. We first recall the following classic result, which we adapt from [12, Chapter 3, Theorem 2.7]:

E.9 **Theorem** (Parametric transversality). *Let M, N, P be smooth manifolds, $W \subseteq N$ a submanifold of N, and $F\colon P \times M \to N$ a smooth map. For all p in P, we denote by F_p the map $F(p,-)\colon M \to N$.*

If F is transverse to W, then the set $\{p \in P \mid F(p,-) \pitchfork W\}$ is residual in P. In other words, for generic $p \in P$, the map F_p is transverse to W.

We immediately have the following corollary.

E.10 **Corollary.** *Let $F\colon P \times M \to N$ be a submersion and $W \subseteq N$ a submanifold. If* $\mathrm{codim}_N W = \dim M$, *then for generic $p \in P$ we have* $\dim(F(p,M) \cap W) = 0$. *If*

$\operatorname{codim}_N W > \dim M$*, then for generic* $p \in P$ *we have* $F(p, M) \cap W = \emptyset$.

We will be applying these transversality results to **jet spaces**.

E.11 **Definition** (jets and multijets). Let $n, m, r \geq 0$ be integers. The space of **jets of order** r **from** $\mathbf{R}^m$ **to** $\mathbf{R}^n$ is

$$J^r(\mathbf{R}^m, \mathbf{R}^n) := \mathbf{R}^m \oplus \bigoplus_{\ell=0}^{r} (\mathscr{P}_{\ell,m})^n,$$

where $\mathscr{P}_{\ell,m}$ is the space of homogeneous polynomials of degree ℓ in m variables.

Let $f\colon \mathbf{R}^m \to \mathbf{R}^n$ be a smooth function and $x \in \mathbf{R}^m$. The **jet of order** r **of** f **at** x is

$$\begin{aligned} j^r(f)(x) &:= \left(\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right)_{\alpha \in \mathbf{N}^m, |\alpha| \leq r} \in J^r(\mathbf{R}^m, \mathbf{R}^n) \\ &= \left(x, f(x), \ldots, \frac{\partial f}{\partial x_i}(x), \ldots, \frac{\partial^r f}{\partial^r x_i}, \ldots \right). \end{aligned}$$

The identification with polynomial spaces comes from identifying the partial derivative $\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)$ with the coefficient in front of the monomial x^α.

Let $k \geq 1$ be an integer. The space of k**-multijets of order** r is just k copies of the space of jets of order r:

$${}_kJ^r(\mathbf{R}^m, \mathbf{R}^n) := J^r(\mathbf{R}^m, \mathbf{R}^n)^k.$$

Similarly, let $f\colon \mathbf{R}^m \to \mathbf{R}^n$ be a smooth function and $\vec{x} := x_1, \ldots, x_k$ distinct points in $\mathbf{R}^m$ (we write $\vec{x} \in \mathbf{R}^{km} \setminus \Delta$, where Δ is the **fat diagonal** of $\mathbf{R}^m$, consisting of all tuples with duplicate points). The k**-multijet of order** r **of** f **at** $x_1, \ldots, x_k$ is

$${}_kj^r(\vec{x}) := (j^r(f)(x_i))_{i=1,\ldots,k}.$$ ■

We study jets through parametric jet maps of the form

$$F\colon \begin{cases} \mathscr{P}_d \times \mathbf{R}^{nk} & \to \quad {}_kJ^r(\mathbf{R}^n, \mathbf{R}) \\ (p, \vec{x}) & \mapsto \quad {}_kj^r(p)(\vec{x}). \end{cases} \tag{E.2}$$

The next result is a key technical ingredient for the rest of the paper.

E.12 **Theorem.** *Let $\mathscr{P}_d$ be the space of real polynomials of degree d in $\mathbf{R}^n$. There exists a function $d(n,k,r)$ such that, for all $d \geq d(n,k,r)$, the map $F\colon \mathscr{P}_d \times \mathbf{R}^{nk} \to {}_kJ^r(\mathbf{R}^n, \mathbf{R})$ is a submersion.*

Proof. Fix d a polynomial degree, $p \in \mathscr{P}_d$ a polynomial, and $\vec{x} := (x_1, \dots, x_k) \in \mathbf{R}^{nk}$ points in $\mathbf{R}^n$. The differential of F at $(p, \vec{x})$ is the linear map

$$DF_{p,\vec{x}}\colon \left\{ \begin{array}{ccc} T_{p,\vec{x}}(\mathscr{P}_d \times \mathbf{R}^{nk}) & \to & T_{F(p,\vec{x})}({}_kJ^r(\mathbf{R}^n, \mathbf{R})) \\ (\dot{p}, \dot{\vec{x}}) & \mapsto & \begin{array}{c} (\dot{\vec{x}}, \dots, \dot{p}(x_i) + \nabla p(x_i)^\top \dot{x}_i, \dots, \\ \dots, \nabla \dot{p}(x_i) + \nabla^2 p(x_i)^\top \dot{x}_i, \dots), \end{array} \end{array} \right.$$

where $T_{p,\vec{x}}(\mathscr{P}_d \times \mathbf{R}^{nk}) = \mathscr{P}_d \times \mathbf{R}^{nk}$ and $T_{F(p,\vec{x})}({}_kJ^r(\mathbf{R}^n, \mathbf{R})) = \mathbf{R}^{nk} \oplus \bigoplus_{\ell=0}^r (\mathscr{P}_{\ell,n})^k$.

Let $(\dot{\vec{z}}, \dot{U}) = (\dot{\vec{z}}, \dot{\vec{u}}^{(1)}, \dots, \dot{\vec{u}}^{(r)}) \in \mathbf{R}^{nk} \oplus \bigoplus_{\ell=1}^r (\mathscr{P}_{\ell,n})^k$. For F to be a submersion at $(p, \vec{x})$, it suffices to find $(\dot{p}, \dot{\vec{x}})$ such that $DF_{p,\vec{x}}(\dot{p}, \dot{\vec{x}}) = (\dot{\vec{z}}, \dot{U})$. This is linear in the coefficients of $\dot{p}$ and in monomials in $\dot{\vec{x}}$. We find that $\dot{\vec{x}} = \dot{\vec{z}}$, and then there are $k\sum_{\ell=0}^r \binom{n}{\ell}$ equations in the coefficients of $\dot{p}$. Since the x_i are pairwise distinct points in $\mathbf{R}^n$, these equations are independent. For there to be a solution in general, it suffices that

$$\dim \mathscr{P}_d = \binom{n+d}{d} \geq k \sum_{\ell=0}^r \binom{n}{\ell}.$$

We set $d(n,k,r)$ to be the smallest d satisfying this inequality. □

E.13 **Corollary.** *Let r_1 and r_2 be jet orders, and $L \geq 0$ an integer. For all $d_1 \geq d(n,k,r_1)$ and $d_2 \geq d(n,k,r_2)$, the map*

$$F\colon \left\{ \begin{array}{ccc} \begin{array}{c} \mathscr{C}_{d_1}^\ell \times \mathscr{C}_{d_2}^m \times \mathbf{R}^n \times \\ (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L \end{array} & \to & J^{r_1}(\mathbf{R}^n, \mathbf{R}^\ell) \times {}_kJ^{r_2}(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L \\ (\vec{p}, \vec{q}, x, \vec{y}, \vec{\kappa}) & \mapsto & (j^{r_1}(\vec{p})(\vec{x}), {}_kj^{r_2}(\vec{q})(\vec{y}), \vec{\kappa}) \end{array} \right.$$

is a submersion.

Proof. We follow the same reasoning as in the proof of Theorem E.12: we want to solve an equation of the form

$$DF_\bullet(\dot{\vec{p}}, \dot{\vec{q}}, \dot{x}, \dot{\vec{y}}, \dot{\vec{\kappa}}) = \begin{array}{ccccccc} (& \dot{z}, \dot{\vec{u}}, & & \dot{\vec{w}}, \dot{V}, & & \dot{\vec{t}} &) \\ \in & T_\bullet J^{r_1}(\mathbf{R}^n, \mathbf{R}^\ell) & \oplus & T_{\bullet k}J^{r_2}(\mathbf{R}^n, \mathbf{R}^m) & \oplus & T_\bullet \mathbf{R}^L, & \end{array}$$

where $\bullet = (\vec{p},\vec{q},x,\vec{y},\vec{\kappa})$ is a point of the codomain and $(\dot{\vec{p}},\dot{\vec{q}},\dot{x},\dot{\vec{y}},\dot{\vec{\kappa}})$ are the unknowns. As before, we first find $\dot{x} = \dot{z}$ and $\dot{\vec{y}} = \dot{\vec{w}}$. Moreover, we have $\dot{\vec{\kappa}} = \dot{\vec{t}}$. The remaining equations each only depend on the coefficients of a single polynomial $\dot{p}_i$ or $\dot{q}_j$, and are identical to those of Theorem E.12. Therefore, since we assume $d_1 \geq d(n,k,r_1)$ and $d_2 \geq d(n,k,r_2)$, we know that we can solve for $\dot{\vec{p}}$ and $\dot{\vec{q}}$. We conclude that F is a submersion. □

E.14 **Example.** If $k \leq n+1$ and $r = 1$, then we can take $d(n,k,r) = 3$. Indeed, for all $n \geq 1$, $\binom{n+3}{3} \geq (n+1)^2 \geq k(\binom{n}{0} + \binom{n}{1})$.

If $k \leq n+1$ and $r = 2$, then we can take $d(n,k,r) = 4$. Indeed, we compute

$$\begin{aligned}
\dim \mathscr{P}_d - k\sum_{\ell=0}^{r}\binom{n}{\ell} &\geq \dim \mathscr{P}_d - (n+1)\sum_{\ell=0}^{r}\binom{n}{\ell} \\
&= \binom{n+4}{4} - (n+1)(\binom{n}{0} + \binom{n}{1} + \binom{n}{2}) \\
&= \frac{(n+1)(n+2)(n+3)(n+4)}{24} - (n+1)(1+n+\frac{n(n-1)}{2}) \\
&= \frac{n+1}{24}((n+2)(n+3)(n+4) - 12(n^2+n+2)) \\
&= \frac{n+1}{24}(n^3 - 3n^2 + 14n). \qquad \text{(E.3)}
\end{aligned}$$

We determine the sign of (E.3) by looking at the derivative of the last factor: $3n^2 - 6n + 14$ has discriminant $36 - 4\cdot 3\cdot 14 < 0$ and value $14 > 0$ at $n = 0$, so the derivative is always positive. At $n = 1$, we have $n^3 - 3n^2 + 14n = 12$, so for all $n \geq 1$, (E.3) is positive. ■

E.3 Genericity in the algebraic setting

Throughout this section, we assume $n \geq 1$. We consider $\ell + m$ real n-variable polynomials $p_1,\dots,p_\ell,q_1,\dots,q_m$ and two algebraic sets $X := V(p_1,\dots,p_\ell)$ and $Y := V(q_1,\dots,q_m)$.

E.3.1 Setup and preliminary genericities

E.15 **Notation.** For $d \in \mathbf{N}$, we denote by $\mathscr{P}_d$ the set of polynomials on $\mathbf{R}^n$ of degree at most d.

We will be handling many tuples of functions and vectors. Given functions $\vec{p} = p_1, \ldots, p_\ell$ and vectors $\vec{x} = x_1, \ldots, x_k$, we denote by $\vec{p}(\vec{x})$ the tuple

$$(p_1(x_1), \ldots, p_1(x_k), p_2(x_1), \ldots, p_2(x_k), \ldots, p_\ell(x_1), \ldots, p_\ell(x_k)).$$

We will also encounter doubly indexed lists of numbers; these are notated with uppercase symbols, e.g. $\Xi = (\xi_{ij})$. We also encounter double indexed lists of matrices; these are notated with the blackboard font, e.g. $\mathbb{H} = (H_{ij})$. ■

E.16 **Definition.** Let $m \in \{1, \ldots, n\}$ and $d \geq 0$. We consider the set of complete intersections in $\mathbf{R}^n$ of codimension m whose defining polynomials all have degree at most d. Denote by $\mathscr{C}_d^m$ the open subset of $(\mathscr{P}_d)^m$ whose elements generate such complete intersections.

To prove results for generic complete intersections, it then suffices to prove the result for a generic element of $\mathscr{C}_d^m$, for all m and d. ■

E.3.2 Finite number of closest points

In this section we prove an analogue of [16, Proposition 3], showing that generically the function m_Y is bounded by $n+1$.

E.17 **Theorem** (cf. [16, Proposition 3]). *For the generic $\vec{q} \in (\mathscr{P}_d)^m$ with $d \geq 2$, writing $Y = V(\vec{q})$, for every $x \in \mathbf{R}^n$ the set $B(x, \operatorname{dist}_Y(x)) \cap Y$ is a nondegenerate simplex with at most $n+1$ points.*

Proof. For all $k \in \{0, \ldots, n+1\}$, define the parameterized jet map

$$F_k \colon \begin{cases} (\mathscr{P}_d)^m \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) & \to \quad {}_{k+1}J^0(\mathbf{R}^n, \mathbf{R}^{m+1}) \\ (\vec{q}, x, \vec{y}) & \mapsto \quad (\vec{y}, \vec{q}(\vec{y}), \|x - \vec{y}\|^2) \end{cases}$$

induced by the function $y \mapsto (\vec{q}(y), \|x-y\|^2)$. Its differential at a point $(\vec{q}, x, \vec{y})$ is

$$DF_{k,(\vec{q},x,\vec{y})} \colon \begin{cases} (\mathscr{P}_d)^m \times \mathbf{R}^n \times \mathbf{R}^{n(k+1)} & \to \quad \mathbf{R}^{n(k+1)} \times \mathbf{R}^{(k+1)m} \times \mathbf{R}^{k+1} \\ (\dot{\vec{q}}, \dot{x}, \dot{\vec{y}}) & \mapsto \quad \begin{array}{c}(\dot{\vec{y}}, \ldots, \dot{q}_j(y_i) + \nabla q_j(y_i)^\top \dot{y}_i, \ldots, \\ \ldots, 2(x - y_i)^\top (\dot{x} - \dot{y}_i), \ldots).\end{array} \end{cases}$$

Let

$$W_k :=$$

$$\left\{ (\vec{z}, S, \vec{t}) \in {}_{k+1}J^0(\mathbf{R}^n, \mathbf{R}^{m+1}) \;\middle|\; \begin{array}{l} \forall i \in \{0,\dots,k\},\ \forall j \in \{1,\dots,m\},\ s_{ij} = 0, \\ \forall i \in \{1,\dots,k\},\ t_i = t_0 \end{array} \right\}$$

be a subspace of ${}_{k+1}J^0(\mathbf{R}^n, \mathbf{R}^{m+1})$ and let us prove that F_k is transverse to W_k. Let $(\vec{q}, x, \vec{y})$ be a point in $(\mathscr{P}_d)^m \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \smallsetminus \Delta)$ such that $F_k(\vec{q}, x, \vec{y}) \in W_k$, and $(\vec{\dot{z}}, \dot{S}, \vec{\dot{t}})$ a tangent vector in $T_{F_k(\vec{q},x,\vec{y})\,k+1}J^0(\mathbf{R}^n, \mathbf{R}^{m+1})$. We are looking for $(\vec{\dot{q}}, \dot{x}, \vec{\dot{y}}) \in T_{(\vec{q},x,\vec{y})}((\mathscr{P}_d)^m \times \mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \smallsetminus \Delta))$ and $(\vec{w}, 0, 0) \in T_{F_k(\vec{q},x,\vec{y})} W_k$ such that

$$\begin{aligned} (\vec{\dot{z}}, \dot{S}, \vec{\dot{t}}) &= DF_{k,(\vec{q},x,\vec{y})}(\vec{\dot{q}}, \dot{x}, \vec{\dot{y}}) + (\vec{w}, 0, 0) \\ &= (\vec{\dot{y}} + \vec{w}, \dots, \dot{q}_j(y_i) + \nabla q_j(y_i)^\top \dot{y}_i, \dots, 2(x - y_i)^\top (\dot{x} - \dot{y}_i), \dots). \end{aligned}$$

Firstly, we observe that x is different from all the y_i. Indeed, if not, then we would have $\|x - y_i\|^2 = 0$ for all $i \in \{0, \dots, k\}$, which contradicts the fact that the y_i are distinct. So we can set $\dot{x} = 0$ and solve the equations $2(x - y_i)^\top (\dot{x} - \dot{y}_i) = \dot{t}_i$ to obtain the $\dot{y}_i$. Next, since $d \geq 2$, we can find each $\dot{q}_j$ by solving the equations $\dot{q}_j(y_i) + \nabla q_j(y_i)^\top \dot{y}_i = \dot{s}_{ij}$. Indeed, we have $k+1$ linear equations in the coefficients of $\dot{q}_j$ and $\dim \mathscr{P}_d = \binom{n+d}{d} \geq \binom{n+2}{2} = \frac{(n+2)(n+1)}{2} \geq k+1$, so a solution exists. Finally, we choose $\vec{w} = \vec{\dot{z}} - \vec{\dot{y}}$ and conclude that F_k is transverse to W_k.

By the parametric transversality theorem (Theorem E.9), for generic $\vec{q}$, $F_k(\vec{q}, -)$ is transverse to W_k. Fix generic $\vec{q}$, let $x \in \mathbf{R}^n$, and take $k+1$ points $y_0, \dots, y_k$ from $B(x, \operatorname{dist}_Y(x)) \cap Y$. Then $F_k(\vec{q}, x, \vec{y}) \in W_k$, so we can write the transversality property:

$$\operatorname{im}(DF_k(\vec{q}, -)_{x,\vec{y}}) + T_{F_k(\vec{q},x,\vec{y})} W_k = T_{F_k(\vec{q},x,\vec{y})\,k+1}J^0(\mathbf{R}^n, \mathbf{R}^{m+1}),$$

or more explicitly,

$$\operatorname{im}(DF_k(\vec{q}, -)_{x,\vec{y}}) + \mathbf{R}^{n(k+1)} \times 0^{(k+1)m} \times \mathbf{R}(1, \dots, 1) = \mathbf{R}^{n(k+1)} \times \mathbf{R}^{(k+1)m} \times \mathbf{R}^{k+1}.$$

Using the expression of DF_k above, we know that

$$DF_k(\vec{q}, -)_{x,\vec{y}} \colon (\dot{x}, \vec{\dot{y}}) \mapsto (\vec{\dot{y}}, \dots, \nabla q_j(y_i)^\top \dot{y}_i, \dots, 2(x - y_i)^\top (\dot{x} - \dot{y}_i), \dots).$$

In particular, when the terms $\nabla q_j(y_i)^\top \dot{y}_i$ equal 0, then we also have $(x - y_i)^\top \dot{y}_i = 0$ because $x - y_i$ and $\nabla q_j(y_i)$ are collinear. Thus we need $((x - y_i)^\top \dot{x})_i$

to span a supplementary space to $\mathbf{R}(1,\ldots,1)$ (by a dimension argument). For this last part to be satisfied, we need the vectors

$$\{(x-y_i)-(x-y_0)\}_{i=1,\ldots,k} = \{y_i - y_0\}_{i=1,\ldots,k}$$

to be linearly independent. This is equivalent to $y_0,\ldots,y_k$ forming a nondegenerate simplex in $\mathbf{R}^n$.

Finally, we pick $\vec{q}$ transverse to every W_k for $k \in \{1,\ldots,n+1\}$: this is a generic choice. By the above arguments, any distinct set of at most $n+2$ points in $B(x,\mathrm{dist}_Y(x)) \cap Y$ forms a nondegenerate simplex. We conclude that the set $B(x,\mathrm{dist}_Y(x)) \cap Y$ has at most $n+1$ points (any more would lead to a contradiction with nondegeneracy) and that these points form a nondegenerate simplex. □

E.3.3 Finite number of critical points

In this section we study the structure of critical points for the function $\mathrm{dist}_Y|_X$. First, we define an indexing for the critical points of $\mathrm{dist}_Y|_X$ without explicitly assuming that the function is a continuous selection, nor that the critical points are nondegenerate, which was the case in Definition E.8.

E.18 **Definition** (k-critical points). Let X be a subset of $\mathbf{R}^n$ and Y a closed definable subset of $\mathbf{R}^n$. For all $k \geq 0$, we define the **k-critical points of** $\mathrm{dist}_Y|_X$ as the critical points x in $X \smallsetminus Y$ of $\mathrm{dist}_Y|_X$ such that $\#(B(x,\mathrm{dist}_Y(x)) \cap Y) = k+1$. We denote by $C_{k,*}(\mathrm{dist}_Y|_X)$ the set of k-critical points of $\mathrm{dist}_Y|_X$. ■

As in Definition E.8, we exclude the case $x \in X \cap Y$. By Theorem E.17, every critical point not in Y is a k-critical point for some $k \in \{0,\ldots,n\}$. Moreover, when x is a nondegenerate critical point, in the sense of Definition E.8, then k is equal to the piecewise linear index $k(x)$ of x.

E.19 **Definition** (Algebraic critical points). Let $d \geq 0$, $\vec{p} \in (\mathscr{P}_d)^\ell$, $\vec{q} \in (\mathscr{P}_d)^m$, and

$k \geq 1$. We define the algebraic set

$$C_k(\vec{p},\vec{q}) \coloneqq \left\{ \begin{array}{c|l} x \in \mathbf{R}^n, & x = \sum_{j=1}^{\ell} \mu_j \nabla p_j(x) + \sum_{i=0}^{k} \lambda_i y_i, \\ y_0,\dots,y_k \in \mathbf{R}^{n(k+1)}, & \sum_{i=1}^{k} \lambda_i = 1, \\ \lambda_0,\dots,\lambda_k \in \mathbf{R}^{k+1}, & \forall j \in \{1,\dots,\ell\},\ p_j(x) = 0, \\ \mu_1,\dots,\mu_\ell \in \mathbf{R}^{\ell}, & \forall i \in \{0,\dots,k\},\ \forall j \in \{1,\dots,m\},\ q_j(y_i) = 0, \\ \vec{\xi}_0,\dots,\vec{\xi}_k \in \mathbf{R}^{m(k+1)}, & \forall i \in \{0,\dots,k\},\ \|x - y_i\|^2 = r^2, \\ r \in \mathbf{R} & \forall i \in \{0,\dots,k\},\ \sum_{j=1}^{m} \xi_{ij} \nabla q_j(y_i) = x - y_i \end{array} \right\}.$$

We then define the **algebraic k-critical points w.r.t. $\vec{p}$ and $\vec{q}$** as the set of points $x \in \mathbf{R}^n$ in the projection of $C_k(\vec{p},\vec{q})$ on the first component. ■

E.20 **Lemma.** *For generic $\vec{p} \in \mathscr{C}_{d_1}^{\ell}$ and $\vec{q} \in \mathscr{C}_{d_2}^{m}$, every critical point x of $\mathrm{dist}_Y|_X$ not in Y is an algebraic k-critical point for some $k \in \{0,\dots,n\}$.*

Proof. Let x be an element of $X \setminus Y$. By Proposition E.2, we have

$$\partial_x f = \mathrm{proj}_{T_x X}\left(\mathrm{conv}\left\{ \frac{x-y}{\|x-y\|} \;\middle|\; y \in B(x, \mathrm{dist}_Y(x)) \cap Y \right\} \right).$$

Recall that x is a critical point if 0 belongs to this set. By Theorem E.17, for generic $\vec{p}$ and $\vec{q}$ in the open subsets $\mathscr{C}_{d_1}^{\ell} \subseteq (\mathscr{P}_{d_1})^{\ell}$ and $\mathscr{C}_{d_2}^{m} \subseteq (\mathscr{P}_{d_2})^{m}$, respectively, the set $B(x, \mathrm{dist}_Y(x)) \cap Y$ is finite with at most $n+1$ elements. Moreover, for each point y in $B(x, \mathrm{dist}_Y(x)) \cap Y$, the function $y \mapsto \|x-y\|$ reaches a local minimum at x, so $x - y$ belongs to the subspace spanned by the $\nabla q_j(y)$.

Therefore, if x is a critical point of $\mathrm{dist}_Y|_X$, then there exist $k \in \{0,\dots,n\}$, $y_0,\dots,y_k$ in Y, $\lambda_0,\dots,\lambda_k$ in $[0,1]$, $\mu_1,\dots,\mu_\ell$ in $\mathbf{R}$, and $\Xi = (\xi_{ij})_{\substack{0\le i\le k\\ 1\le j\le m}}$ in $\mathbf{R}^{(k+1)\times m}$ such that

$$\begin{gathered} \sum_{i=0}^{k} \lambda_i = 1, \\ \sum_{i=0}^{k} \lambda_i (x - y_i) = \sum_{j=1}^{\ell} \mu_i \nabla p_i(x), \\ \forall i \in \{1,\dots,k\},\ \|x - y_i\| = \|x - y_0\|, \\ \forall i \in \{0,\dots,k\},\ \sum_{j=1}^{m} \xi_{ij} \nabla q_j(y_i) = x - y_i, \end{gathered} \tag{E.4}$$

and so we conclude that $(x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, \mathrm{dist}_Y(x)) \in C_k(\vec{p}, \vec{q})$ and x is an algebraic k-critical point. $\square$

E.21 **Theorem** (Finite critical points). *For $d_1, d_2 \geq 3$ and $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, let $X = V(\vec{p})$ and $Y = V(\vec{q})$. Then, for the generic choice of $\vec{p}$ and $\vec{q}$, there are only a finite number of critical points of $\mathrm{dist}_Y|_X$ not in Y. Moreover, for every $k \in \{0, \ldots, n\}$, the number of k-critical points of $\mathrm{dist}_Y|_X$ is bounded by*

$$\#C_{k,*}(\mathrm{dist}_Y|_X) \leq c(k, \ell, m, n) d_1^n d_2^{n(k+1)},$$

for some $c(k, \ell, m, n) > 0$ depending on k, ℓ, m, and n only.

Proof. Consider the map

$$F^1 : \begin{cases} \begin{array}{c} \mathscr{C}^{\ell}_{d_1} \times \mathscr{C}^{m}_{d_2} \times \mathbf{R}^n \times \\ (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L \end{array} & \longrightarrow & J^1(\mathbf{R}^n, \mathbf{R}^{\ell}) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L \\ (\vec{p}, \vec{q}, x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, r) & \mapsto & (x, \vec{p}(x), \nabla\vec{p}(x), \vec{y}, \vec{q}(\vec{y}), \nabla\vec{q}(\vec{y}), \vec{\lambda}, \vec{\mu}, \Xi, r), \end{cases}$$

where $L = (k+1) + \ell + (k+1)m + 1 = k + \ell + m + km + 2$. Consider the following subspace of $J^1(\mathbf{R}^n, \mathbf{R}^{\ell}) \times {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L$:

$$W^1 := \left\{ \begin{array}{r|l} (x, \vec{s}, \vec{u}) \in J^1(\mathbf{R}^n, \mathbf{R}^{\ell}), & x = \sum_{j=1}^{\ell} \mu_j u_j + \sum_{i=0}^{k} \lambda_i y_i, \\ (\vec{y}, T, V) \in {}_{k+1}J^1(\mathbf{R}^n, \mathbf{R}^m), & \sum_{i=0}^{k} \lambda_i = 1, \\ \vec{\lambda} \in \mathbf{R}^{k+1}, & \forall j \in \{1, \ldots, \ell\},\ s_i = 0, \\ \vec{\mu} \in \mathbf{R}^{\ell}, & \forall i \in \{0, \ldots, k\},\ \forall j \in \{1, \ldots, m\},\ t_{ij} = 0, \\ \Xi \in \mathbf{R}^{(k+1) \times m}, & \forall i \in \{0, \ldots, k\},\ \|x - y_i\|^2 = r^2, \\ r \in \mathbf{R} \sum_i & \forall i \in \{0, \ldots, k\},\ \sum_{j=1}^{m} \xi_{ij} v_{ij} = x - y_i \end{array} \right\}.$$

We observe that the projection of $\mathrm{im}\, F^1(\vec{p}, \vec{q}, -) \cap W^1$ onto the component x contains the set of algebraic k-critical points w.r.t. $\vec{p}$ and $\vec{q}$. Checking the equations in the definition of W^1 from the bottom up, we see that each new equation introduces a new variable. Thus the equations are independent, and so W^1 has codimension

$$\begin{aligned} \mathrm{codim}\, W^1 &= (k+1)n + (k+1) + (k+1)m + \ell + 1 + n \\ &= k + \ell + m + 2n + km + kn + 2 \end{aligned}$$

$$
\begin{aligned}
&= 2n + kn + L \\
&= \dim(\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L).
\end{aligned}
$$

By Corollary E.13 and Example E.14, for $d_1, d_2 \geq 3$, the first two summands are submersions, and so F^1 is as well. By Corollary E.10, for generic $\vec{p}, \vec{q}$ the set of algebraic k-critical points is an algebraic set of dimension 0, so it is finite. By Lemma E.20, we conclude that the number of critical points is finite. In fact, we have shown that the set of algebraic critical points $C_k(\vec{p}, \vec{q})$ is finite.

In order to estimate the cardinality of $C_{k,*}(\mathrm{dist}_Y|_X)$ under the assumptions of the statement, we observe that this set is the projection on the x-coordinate of the set $\Lambda_k \subseteq \mathbf{R}^n \times \mathbf{R}^{n(k+1)} \times \mathbf{R}^{k+1} \times \mathbf{R}^{\ell} \times \mathbf{R}^{m(k+1)}$ consisting of the points $(x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi)$ solving the system (E.4) with $x \in V(\vec{p})$, $y_i \in V(\vec{q})$ for all $i \in \{0, \dots, k\}$. Since $C_{k,*}(\mathrm{dist}_Y|_X)$ is finite (recall that we are excluding the critical points in $X \cap Y$), its cardinality equals the number of its connected components; moreover each component of Λ_k projects to a component of $C_{k,*}(\mathrm{dist}_Y|_X)$ and we have

$$
\# C_{k,*}(\mathrm{dist}_Y|_X) = b_0(C_{k,*}(\mathrm{dist}_Y|_X)) \leq b_0(\Lambda_k) \leq b(\Lambda_k),
$$

where $b(\Lambda_k)$ denotes the sum of the Betti numbers of Λ_k. In order to control this number, we use [3, Theorem 2.6], which states that the sum of the Betti numbers of an algebraic set $V \subseteq \mathbf{R}^{n_1 + \cdots + n_a}$ defined by some polynomials $p_j(x_1, \dots, x_a)$ of degree at most c_i[1] in the variable $x_i \in \mathbf{R}^{n_i}$ can be bounded by

$$
b(V) \leq O(1)^{n_1 + \cdots + n_a} a^{3(n_1 + \cdots + n_a)} c_1^{n_1} \cdots c_a^{n_a}.
$$

In our case, the defining polynomials are $\vec{p}$, $\vec{q}$, $\sum \lambda_i - 1$, $\sum \lambda_i (x - y_i) - \sum \mu_i \nabla p_i(x)$, $\|x - y_i\|^2 - \|x - y_0\|^2$, and $\sum_{j=1}^m \xi_{ij} \nabla q_j(y_i) - (x - y_i)$. We therefore have $\vec{n} = (n, n(k+1), k+1, \ell, m(k+1))$ and $\vec{c} = (d_1, d_2, 1, 1, 1)$, so that

$$
\begin{aligned}
b(\Lambda_k) &\leq O(1)^{n + n(k+1) + (k+1) + \ell + m(k+1)} d_1^n d_2^{n(k+1)} \\
&\leq c(k, \ell, m, n) d_1^n d_2^{n(k+1)}. \qquad \square
\end{aligned}
$$

E.22 **Remark.** The same proof works in the case $(X, Y) = (\mathbf{R}^n, V(\vec{q}))$ (see also Remark E.30), with $\vec{c} = (1, d, 1, 1, 1)$: for generic polynomials $\vec{q} \in \mathscr{C}_d^m$, the function

[1] In the original statement, the authors require $c_i \geq 2$. Since the c_i's are upper bounds, we implicitly replace our $c_i = 1$ with $c_i = 2$, which then gets included into the constant factor.

$\mathrm{dist}_{V(\vec{q})}$ has only finitely many critical points, all these critical points are algebraic and the number of algebraic k-critical points is bounded by

$$\#C_{k,*}(\mathrm{dist}_{V(q)}) \le \tilde{c}(k,n)d^{n(k+1)}.$$

In particular this recovers and generalizes to all dimensions the bound $\#C_{1,*} \le c(n)d^{2n}$ that one can obtain by combining [10, Corollary 4.5] with the EDD estimate [9, Corollary 2.10]. ■

E.3.4 Continuous selections

E.23 **Proposition.** *Let $x \in \mathbf{R}^n$. For $d_1 \ge 3$ and $d_2 \ge 4$ and generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$, the critical points of $\mathrm{dist}_Y|_X$ are regular values of the normal exponential map of Y.*

Proof. The normal exponential map of Y is

$$\exp\colon \left\{ \begin{array}{ccc} NY \cong Y \times \mathbf{R}^m & \to & \mathbf{R}^n \\ (y,t) & \mapsto & y + \sum_j t_j \nabla q_j(y). \end{array} \right.$$

Its differential at a point $(y,t) \in NY$ is

$$D\exp_{(y,t)}\colon \left\{ \begin{array}{ccc} TNY \subseteq \mathbf{R}^n \times \mathbf{R}^m & \to & T\mathbf{R}^n \cong \mathbf{R}^n \\ (\dot{y},\dot{t}) & \mapsto & \dot{y} + \sum_j (t_j H(q_j)(y)\dot{y} + \dot{t}_j \nabla q_j(y)). \end{array} \right.$$

Thus a point x in $\mathbf{R}^n$ is a critical value of exp if, and only if, it can be written as $x = y + \sum_j t_j \nabla q_j(y)$ for some (y,t) in NY such that the block matrix $[I_n + \sum_j t_j H(q_j)(y), \nabla\vec{q}(y)]$ restricted to $T_yY \times \mathbf{R}^m$ has rank strictly less than n. Consider the map

$$F^2\colon \left\{ \begin{array}{ccc} \begin{array}{c} \mathscr{C}^{\ell}_{d_1} \times \mathscr{C}^{m}_{d_2} \times \mathbf{R}^n \times \\ (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L \end{array} & \to & J^1(\mathbf{R}^n,\mathbf{R}^{\ell}) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}^m) \times \mathbf{R}^L \\ (\vec{p},\vec{q},x,\vec{y},\vec{\lambda},\vec{\mu},\Xi,r) & \mapsto & \begin{array}{c} (x,\vec{p}(x),\nabla\vec{p}(x), \\ \vec{y},\vec{q}(\vec{y}),\nabla\vec{q}(\vec{y}),H(\vec{q})(\vec{y}),\vec{\lambda},\vec{\mu},\Xi,r), \end{array} \end{array} \right.$$

and let

$$W^2 := \left\{ \begin{array}{l} (x,\vec{s},\vec{u},\vec{y},T,V,\mathbb{H},\vec{\lambda},\vec{\mu},\Xi,r) \in J^1(\mathbf{R}^n,\mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}) \times \mathbf{R}^L : \\ (x,\vec{s},\vec{u},\vec{y},T,V,\vec{\lambda},\vec{\mu},\Xi r) \in W^1 \end{array} \right\},$$

where W^1 is defined in the proof of Theorem E.21. We define the subspace of $J^1(\mathbf{R}^n,\mathbf{R}^\ell) \times {}_{k+1}J^2(\mathbf{R}^n,\mathbf{R}^m) \times \mathbf{R}^L$

$$W^2_{\text{crit}} := \left\{ \begin{array}{l} (x,\vec{s},\vec{u},\vec{y},T,V,\mathbb{H},\vec{\lambda},\vec{\mu},\Xi,r) \in W^2 : \\ \forall i \in \{0,\dots,k\},\ \operatorname{rank}[(I_n + r\sum_j H_{ij})(I_n - V_iV_i^\top), V_i] < n \end{array} \right\}.$$

This space is defined by the same equations as W^1 and additional equations that involve the H_i. These new equations are independent of those defining W^2. Indeed, given $(x,\vec{s},\vec{u},\vec{y},T,V,\mathbb{H},\vec{\lambda},\vec{\mu},\Xi,r)$ in W^2 and $i \in \{0,\dots,k\}$, we can assume $H_{ij} = 0$, which gives us $\operatorname{rank}[(I_n - V_iV_i^\top), V_i] = n$.

We deduce that W^2_{crit} has codimension strictly greater than that of W^1, which is equal to the dimension of $\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L$. By Corollary E.13 and Example E.14, for $d_1 \geq 3$ and $d_2 \geq 4$, the map F^2 is a submersion. By Corollary E.10, for generic $\vec{p}$ and $\vec{q}$, the critical points of $\operatorname{dist}_Y|_X$ miss W^2_{crit}. We conclude that, for generic $\vec{p}$ and $\vec{q}$, the critical points of $\operatorname{dist}_Y|_X$ are regular values for the normal exponential map of Y. □

E.24 **Corollary.** *For generic $\vec{p} \in \mathscr{C}^\ell_{d_1}$ and $\vec{q} \in \mathscr{C}^m_{d_2}$ of respective degrees $d_1 \geq 3$ and $d_2 \geq 4$, denoting by $X = V(\vec{p})$ and $Y = V(\vec{q})$ the function $\operatorname{dist}_Y|_X$ is a continuous selection of $\mathscr{C}^2$ functions around its critical points.*

Proof. By Proposition E.23, for generic $\vec{p}$ and $\vec{q}$, the critical points of $\operatorname{dist}_Y|_X$ are regular values of the normal exponential map of Y. By Proposition E.7, we deduce that $\operatorname{dist}_Y|_X$ is a $\mathscr{C}^2$ selection around its critical points. □

E.3.5 Nondegeneracy of critical points

Now we describe the degeneracy of points algebraically. To do this, we need to describe the Hessian of $\operatorname{dist}_Y|_X$ at x in terms of the derivatives of $\vec{p}$ and $\vec{q}$.

For all y in Y, we define $g_y := \operatorname{dist}_{B(y,\varepsilon)\cap Y}$ and $f_y := g_y\,|_X$ for $y \in Y$. We also introduce the following notation: for a symmetric matrix $H \in \operatorname{Sym}(n,\mathbf{R})$ and a subspace $V \subseteq \mathbf{R}^n$, we denote by $H\,|_V$ a matrix representing the restriction of the quadratic form $h(x) = x^\top Hx$ to V. If $U = [v_1,\dots,v_\ell] \in \mathbf{R}^{n\times\ell}$ is a matrix whose columns form a basis for V, then $U(U^\top U)^{-1}U^\top$ is the orthogonal projection matrix onto V, and so one can take

$$H\,|_V = (U(U^\top U)^{-1}U^\top)H(U(U^\top U)^{-1}U^\top).$$

E.25 **Definition** (Algebraic degenerate k-critical points). The set of **algebraic degenerate k-critical points w.r.t. $\vec{p}$ and $\vec{q}$** is the projection of the set

$$D_k(\vec{p},\vec{q}) := \tag{E.5}$$

$$\left\{ \begin{array}{l} (x,\vec{y},\vec{\lambda},\vec{\mu},\Xi,r) \in C_k(\vec{p},\vec{q}) : \\ \det\left(\sum_{j=1}^{\ell} \mu_j H(p_j)(x)|_{\partial_x f^\perp} + r\sum_{i=0}^{k} \lambda_i H(g_{y_i})(x)|_{\partial_x f^\perp}\right) = 0 \end{array} \right\} \tag{E.6}$$

onto the first component x, where $C_k(\vec{p},\vec{q})$ is defined in Definition E.19. ■

The use of the adjective "algebraic" is justified by the following lemma.

E.26 **Lemma.** *For all $k \in \{1,\dots,n\}$, the set $D_k(\vec{p},\vec{q})$ is algebraic.*

Proof. Let x be an element of $\mathbf{R}^n \setminus Y$ and y a closest point in Y at distance $r = \operatorname{dist}(x,y)$. We can decompose $\mathbf{R}^n$ as $T_yY \oplus \mathbf{R}(x-y) \oplus S$, where S is the subspace of N_yY orthogonal to $\mathbf{R}(x-y)$. By the discussion after [14, Proposition 3.3], for all $s \in (0,r]$, the Hessian of g_y at $sx+(r-s)y$ is diagonalizable in a common basis, with an eigenvalue 0 corresponding to the eigenvector $v := x-y$, $(m-1)$ eigenvalues $\frac{1}{s}$ corresponding to the eigenvectors in S, and $n-m$ eigenvalues $\beta_1(s),\dots,\beta_{n-m}(s)$ corresponding to eigenvectors in T_yY. Moreover, the $\beta_i(s)$ are bounded and satisfy the differential equation

$$\beta_i'(s) = -\beta_i(s)^2,$$

which we solve as $\beta_i(s) = \frac{\beta_i(0)}{1+s\beta_i(0)}$, where $\beta_i(0)$ is the limit of $\beta(s)$ at 0. Note that $\beta_i(0)$ is (up to sign) the curvature of Y at y in the direction of the associated eigenvalue. Therefore, denoting by H the Hessian of the $g_y|_{T_yY}$ at y, the Hessian of g_y at x is

$$H(g_y)(x) = \left(\frac{H}{I_n + rH}\right)|_{T_yY} \oplus \frac{1}{r} I_n|_S \oplus 0|_{\mathbf{R}(x-y)}.$$

Next, we observe that the Hessian H is nothing but the second fundamental form of Y at y in the direction $(x-y) \in N_yY$. Since Y is a complete intersection, following [5, §6.2], we have

$$H = \frac{1}{r}\sum_{j=1}^{m} \xi_j H(q_j),$$

where the ξ_i are given by $(x-y) = \sum_{i=1}^{m} \xi_i \nabla q_i$.

Finally, since Y is a complete intersection, the columns of the matrix $V := (\nabla q_i(y))_{1\le i\le m}$ form a basis of N_yY. Therefore $Q := V(V^\top V)^{-1}V^\top$ is the orthogonal projection matrix onto N_yY. Thus we get

$$H(g_y)(x)|_{T_yY} = (I_n - Q)\frac{H}{I_n + rH}(I_n - Q).$$

We restrict to T_yY because this is the only relevant part for (E.5).

Now let $(x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, r) \in D_k(\vec{p}, \vec{q})$. Generically, by transversality of the defining hypersurfaces of X and Y, the columns of the matrix $U := (\nabla\vec{p}(x), \nabla\vec{q}(\vec{y}))$ are linearly independent. We are interested in the projection of the Hessians $H(p_j)(x)$ and $H(g_{y_i})(x)$ onto $T_xX \cap (\bigcap_{i=0}^k T_{y_i}Y)$. This projection is $L := (I_n - U(U^\top U)^{-1}U)$. The equation in (E.5) then becomes

$$\begin{aligned}&\operatorname{rank}\left(\sum_{j=1}^{\ell}\mu_j L^\top H(p_j)(x)L + r\sum_{i=0}^{k}\lambda_i L^\top \frac{\sum_{j=1}^m \xi_j H(q_j)(y_i)}{rI_n + \sum_{j=1}^m \xi_j H(q_j)(y_i)} L\right)\\ &< n - \ell - (k+1)m.\end{aligned}$$

This equation is algebraic, and this concludes the proof. □

E.27 **Remark.** By the above reasoning, the degeneracy condition of (E.5) never holds if $\ell + (k+1)m < n$. Indeed, in this case, condition 2 of Remark E.6 holds automatically because the subspace $T(x)$ is trivial, and therefore (generically) all critical points are nondegenerate. ■

E.28 **Lemma.** *For generic $\vec{p} \in \mathscr{C}_{d_1}^{\ell}$ and $\vec{q} \in \mathscr{C}_{d_2}^{m}$ with $d_1 \ge 3$ and $d_2 \ge 4$, every degenerate critical point of $\operatorname{dist}_Y|_X$ not in Y is an algebraic degenerate k-critical point for some $k \in \{0, \dots, n\}$.*

Proof. We pick generic $\vec{p}$ and $\vec{q}$ such that, for all x in X, the set $B(x, \operatorname{dist}_Y(x)) \cap Y$ is a nondegenerate simplex (Theorem E.17) and that $\operatorname{dist}_Y|_X$ is a $\mathscr{C}^2$ selection around each of its critical points (Corollary E.24). In this context, a critical point is degenerate if, and only if, condition 2 of Remark E.6 holds.

Let x be a critical point of $\operatorname{dist}_Y|_X$. As in the proof of Lemma E.20, there exist $k \in \{0, \dots, n\}$, $y_0, \dots, y_k$ in Y, $\lambda_0, \dots, \lambda_k$ in $[0,1]$, $\mu_1, \dots, \mu_\ell$ in $\mathbf{R}$, and $\Xi = (\xi_{ij})_{\substack{0\le i\le k\\ 1\le j\le m}}$ in $\mathbf{R}^{(k+1)\times m}$ such that $(x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, \operatorname{dist}_Y(x))$ is an element of $C_k(p, q)$.

We can write $\operatorname{dist}_Y|_X$ as the continuous selection of $\mathscr{C}^2$ functions

$$\operatorname{dist}_Y|_X = \min_{i=0,\dots,k} f_{y_i}.$$

Using Remark E.6, we see that the critical point x is degenerate if, and only if, the matrix

$$\begin{aligned}&\sum_{i=0}^{k}\left(\lambda_i H(g_{y_i})(x)\,|_{\partial_x f^\perp} + \lambda_i \sum_{j=1}^{\ell} \mu_{ij} H(p_j)(x)\,|_{\partial_x f^\perp}\right)\\ &= \frac{1}{\operatorname{dist}_Y(x)} \sum_{j=1}^{\ell} \mu_j H(p_j)(x)\,|_{\partial_x f^\perp} + \sum_{i=0}^{k} \lambda_i H(g_{y_i})(x)\,|_{\partial_x f^\perp},\end{aligned}$$

is singular. We conclude that if x is a degenerate critical point of $\operatorname{dist}_Y|_X$, then $(x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, \operatorname{dist}_Y(x))$ is an element of $D_k(\vec{p}, \vec{q})$, and so x is an algebraic degenerate k-critical point with $k \leq n$. □

E.29 **Theorem** (Genericity of nondegeneracy). *For generic $\vec{p} \in \mathscr{C}^{\ell}_{d_1}$ and $\vec{q} \in \mathscr{C}^{m}_{d_2}$ with $d_1, d_2 \geq 4$, all critical points of $\operatorname{dist}_Y|_X$ that are not in Y are nondegenerate.*

Proof. Consider the map

$$F^3 : \left\{ \begin{array}{ccc} \mathscr{C}^{\ell}_{d_1} \times \mathscr{C}^{m}_{d_2} \times \mathbf{R}^n \times & & \\ (\mathbf{R}^{n(k+1)} \smallsetminus \Delta) \times \mathbf{R}^L & \to & J^2(\mathbf{R}^n, \mathbf{R}) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}) \times \mathbf{R}^L \\ & & \\ (\vec{p}, \vec{q}, x, \vec{y}, \vec{\lambda}, \vec{\mu}, \Xi, r) & \mapsto & \begin{array}{c}(x, \vec{p}(x), \nabla\vec{p}(x), H(\vec{p})(x),\\ \vec{y}, \vec{q}(\vec{y}), \nabla\vec{q}(\vec{y}), H(\vec{q})(\vec{y}), \vec{\lambda}, \vec{\mu}, \Xi, r),\end{array} \end{array} \right.$$

and let

$$W^3 := \left\{ \begin{array}{l} (x, \vec{s}, \vec{u}, \vec{G}, \vec{y}, T, V, \mathbb{H}, \vec{\lambda}, \vec{\mu}, \Xi, r) \in J^2(\mathbf{R}^n, \mathbf{R}^\ell) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L : \\ (x, \vec{s}, \vec{u}, \vec{y}, T, V, \vec{\lambda}, \vec{\mu}, \Xi, r) \in W^1 \end{array} \right\},$$

where W^1 is defined in the proof of Theorem E.21.

Let us find a system equations of equations such that, for generic $\vec{p}$ and $\vec{q}$, elements of $F^3(\{\vec{p}\} \times \{\vec{q}\} \times D_k(\vec{p}, \vec{q}))$ satisfy these equations. We define the subspace of $J^2(\mathbf{R}^n, \mathbf{R}^\ell) \times {}_{k+1}J^2(\mathbf{R}^n, \mathbf{R}^m) \times \mathbf{R}^L$

$$W^3_{\deg} := \left\{ \begin{array}{l} (x, \vec{s}, \vec{u}, \vec{G}, \vec{y}, T, V, \mathbb{H}, \vec{\lambda}, \vec{\mu}, \Xi, r) \in W^3 : \\ U = (\vec{u}, V) \in \mathbf{R}^{n \times (\ell + (k+1)m)}, \\ L = I_n - U(U^\top U)^{-1} U^\top \in \mathbf{R}^{n \times n}, \\ \operatorname{rank}\left(\sum_{j=1}^{\ell} \mu_j L^\top G_j L + r \sum_{i=0}^{k} \lambda_i L^\top \frac{\sum_{j=1}^m \xi_j H_{ij}}{r I_n + \sum_{j=1}^m \xi_j H_{ij}} L \right) \\ \quad < n - \ell - (k+1)m \end{array} \right\}.$$

If $\ell + (k+1)m \geq n$, then this last condition is never satisfied, which means that all critical points are nondegenerate, which concludes the proof (see Remark E.27).

We now assume that $\ell + (k+1)m < n$. We need to check that the last inequality is independent of the other equations defining W^3. It suffices to find an element of W^3 such that this rank is $n - \ell - (k+1)m$. Since the equations defining W^3 do not involve $\vec{G}$ nor $\mathbb{H}$, we can choose these terms freely. Let $(x, \vec{s}, \vec{u}, \vec{y}, T, V, \mathbb{H}, \vec{\lambda}, \vec{\mu}, \Xi, r)$ be an element of W^3. Without loss of generality, we assume $\lambda_0 \neq 0$ and $\xi_0 \neq 0$, and we set $G_j = 0$ for all $j \in \{1, \dots, \ell\}$, $H_{ij} = 0$ for all $(i, j) \neq (0, 0)$, and $H_{00} = I_n$. Up to scaling, we can assume that $r \neq -\xi_0$. In this case, we are considering the rank of the matrix

$$\lambda_0 \frac{\xi_0}{r + \xi_0} L,$$

which is $n - \ell - (k+1)m$ by transversality of the polynomials $\vec{p}$ and $\vec{q}$.

We deduce that $W^3_{\deg}$ has codimension strictly greater than that of W^1, which is equal to the dimension of $\mathbf{R}^n \times (\mathbf{R}^{n(k+1)} \setminus \Delta) \times \mathbf{R}^L$. By Corollary E.13 and Example E.14, for $d_1, d_2 \geq 4$, the map F^3 is a submersion. By Corollary E.10, for generic $\vec{p}$ and $\vec{q}$, the image $\operatorname{im} F^3(\vec{p}, \vec{q}, -)$ misses $W^3_{\deg}$. In particular, the image of $D_k(\vec{p}, \vec{q})$ by $F^3(\vec{p}, \vec{q}, -)$ misses $W^3_{\deg}$. By Lemmas E.26 and E.28, we conclude that, for generic $\vec{p}$ and $\vec{q}$, the critical points of $\operatorname{dist}_Y|_X$ are nondegenerate. □

E.30 **Remark.** The proofs throughout Section E.3 do not work as-is for the case $X = \mathbf{R}^n$ because we require $\vec{p}$ to be of degree at least 3. However, the results are still true, as can be verified by running through the proofs of Theorems E.21 and E.29 and Proposition E.23 with the parameters $\vec{p}$, $\vec{\mu}$, $\vec{s}$, $\vec{u}$, and $\vec{G}$ removed from the definition of W^1, W^2, and W^3, and noting that Corollary E.13 can still be applied. ■

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E.A Summary of notations for variables

The following conventions are used for variables in this paper (exceptions may occur):

Symbol	Description
X, Y	subset of $\mathbf{R}^n$
x, y	(real vector) element of X, Y resp.
z	(real vector) element of $\mathbf{R}^n$
f, g	real-valued function
p, q	real-valued polynomial function
d	(integer) polynomial degree
i, j	(integer) index
k, ℓ, m, n	(integer) size of tuple, (co)dimension
$\kappa, \lambda, \mu, \xi$	(real number) linear coefficient
r	(real number) distance, (integer) jet order
s, t	(real number) 0^{th} order derivative
u, v	(real vector) 1^{st} order derivative
G, H	(real matrix) 2^{nd} order derivative
U, V	(real matrix) basis of normal space
L, Q	(real matrix) projection matrix

As explained in Notation E.15, for every symbol a, we can write a tuple as $\vec{a} := (a_1, \dots, a_k)$. For a lower case symbol a, we write a doubly indexed set as $A = (a_{ij})_{ij}$; for an upper case symbol A, we write a double indexed set as $\mathbb{A} = (A_{ij})_{ij}$. In addition, given an element a of a manifold, we denote an element of the tangent space at a by $\dot{a}$.

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